XXX. *On the Dynamical Stability and on the Oscillations of Floating Bodies.*

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If a body be made, by the action of certain disturbing forces, to pass from one position of equilibrium into another, and if in each of the intermediate positions these forces are in excess of the forces opposed to its motion, it is obvious that, by reason of this excess, the motion will be continually accelerated, and that the body will reach its second position with a certain finite velocity, whose effect (measured under the form of *vis viva*) will be to carry it beyond that position. This however passed, the case will be reversed, the resistances will be in excess of the moving forces, and the body's velocity being continually diminished and eventually destroyed, it will, after resting for an instant, again return towards the position of equilibrium through which it had passed. It will not however finally rest in this position until it has completed other oscillations about it. Now the amplitude of the first oscillation of the body beyond the position in which it is finally to rest, being its greatest amplitude of oscillation, involves practically an important condition of its stability; for it may be an amplitude sufficient to carry the body into its next adjacent position of equilibrium, which being, of necessity, a position of unstable equilibrium, the motion will be yet further continued and the body overturned. Different bodies requiring moreover different amounts of work to be done upon them to produce in all the same amplitude of oscillation, that is (relatively to that amplitude) the most stable which requires the greatest amount of work to be so done upon it. It is this condition of stability, dependent upon dynamical considerations, to which, in the following paper, the name of dynamical stability is given.

I cannot find that the question has before been considered in this point of view, but only in that which determines whether any given position be one of stable, unstable or mixed equilibrium; or which determines what pressure is necessary to retain the body at any given inclination from such a position.

1. To the discussion of the conditions of the dynamical stability of a body the principle of *vis viva* readily lends itself. That principle*, when translated into a language which the labours of M. Poncelet have made familiar to the uses of practical science, may be stated as follows:—

* See Poisson, Mécanique, chap. ix. Art. 565; Poncelet, Mécanique Industrielle, *passim*; Mechanical Principles of Engineering by the author of this paper, Art. 129.

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"When, being acted upon by given forces, a body or system of bodies has been moved from a state of rest, the difference between the aggregate *work* of those forces whose tendencies are *in* the directions in which their points of application have been moved, and that of the forces whose tendencies are in the opposite direction, is equal to one-half the *vis viva* of the system."

Thus, if Σu_1 be taken to represent the aggregate work of the forces by which a body has been displaced from a position in which it was at rest, and Σu_2 the aggregate work (during this displacement) of the other forces applied to it; and if the terms which compose Σu_1 and Σu_2 be understood to be taken positively or negatively, according as the tendencies of the corresponding forces are in the directions in which their points of application have been made to move or in the opposite directions; then representing the aggregate *vis viva* of the body by $\frac{1}{g}\Sigma wv^2$

$$\Sigma u_1+\Sigma u_2=\frac{1}{2g}\Sigma wv^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Now Σu_2 representing the aggregate work of those forces which acted upon the body in the position from which it has been moved, may be supposed to be known; Σu_1 may therefore be determined in terms of the *vis viva*, or conversely.

2. In the extreme position into which the body is made to oscillate and from which it begins to return, it, for an instant, *rests*. In this position, therefore, its *vis viva* disappears, and we have

$$\Sigma u_1+\Sigma u_2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

This equation, in which Σu_1 and Σu_2 are functions of the impressed forces and of the inclination, determines the extreme position into which the body is made to roll by the action of given disturbing forces; or, conversely, it determines the forces by which it may be made to roll into a given extreme position.

3. The position in which it will finally *rest* is determined by the *maximum* value of $\Sigma u_1+\Sigma u_2$ in equation 1; for, by a well-known property, the *vis viva* of a system* attains a maximum value when it passes through a position of stable, and a minimum, when it passes through a position of unstable equilibrium. The extreme position into which the body oscillates is therefore essentially different from that in which it will finally rest.

4. Different bodies, requiring different amounts of work to be done upon them to bring them to the same given inclination, *that* is (relatively to that inclination) the most stable, which requires the greatest amount of work to be so done upon it, or in respect to which Σu_1 is the greatest. If, instead of all being brought to the same given inclination, each is brought into a position of unstable equilibrium, the corresponding value of Σu_1 represents the amount of work which must be done upon it to *overthrow* it, and may be considered to measure its *absolute*, as the former value

* Poisson, Mécanique, Art. 571.

measures its *relative* dynamical stability*. The absolute dynamical stability of a body thus measured I propose to represent by the symbol U, and its relative dynamical stability, as to the inclination θ, by $U(\theta)$.

The measure of the absolute dynamical stability of a body is the maximum value of its relative stability, or U the maximum of $U(\theta)$; for whilst the body is made to incline from its position of stable equilibrium, it continually tends to return to it until it passes through a position of unstable equilibrium, when it tends to recede from it; the aggregate amount of work necessary to produce this inclination must therefore continually increase until it passes through that position and afterwards diminish.

5. The work opposed by the weight of a body to any change in its position is measured by the product of the vertical elevation of its centre of gravity by its weight†. Representing therefore by W the weight of the body, and by ΔH the vertical displacement of its centre of gravity when it is made to incline through an angle θ, and observing that the displacement of this point is in a direction opposite to that in which the force applied to it acts, we have $\Sigma u_2 = -W.\Delta H$, and by equation 2,

$$U(\theta) - W.\Delta H = 0. \quad \quad (3.)$$

If therefore no other force than its weight be opposed to a body's being overthrown, its absolute dynamical stability, when resting on a rigid surface, is measured by *the product of its weight by the height through which its centre of gravity must be raised to bring it from a stable into an unstable position of equilibrium.*

6. *The Dynamical Stability of Floating Bodies.*—The action of gusts of wind upon a ship or of blows of the sea being measured in their effects upon it by their work, that vessel is the most stable under the influence of these, or will roll and pitch the least (other things being the same), which requires the greatest amount of work to be done upon it to bring it to a given inclination; or, in respect to which the relative dynamical stability $U(\theta)$ is the greatest for a given value of θ. In another sense, that ship may be said to be the most stable which would require the greatest amount of work to be done upon it to bring it into a position from which it would not again right itself, or whose absolute dynamical stability U is the greatest. Subject to the one condition, the ship will *roll* the least, and subject to the other, it will be the least likely to roll *over*.

Thus the theory of dynamical stability involves a question of naval construction, and it is principally with reference to this question that I have entered on the discussion of it.

* It is obvious that the absolute dynamical stability of a body may be greater than that of another, whilst its stability, relatively to a given inclination, is less; *less* work being required to incline it than the other at that angle, but more, entirely to overthrow it.

† PONCELET, Mécanique Industrielle, 2me partie, Art. 50; MOSELEY, Mechanical Principles of Engineering, Art. 60.

7. Let a body be conceived to float, acted upon by no other forces than its weight W, and the upward pressure of the water (equal to its weight); which forces may be conceived to be applied respectively to the centre of gravity of the body and to the centre of gravity of the displaced fluid; and let it be supposed to be subjected to the action of a third force whose direction is parallel to the surface of the fluid. Let ΔH_1 represent the vertical displacement of the centre of gravity of the body thereby produced*, and ΔH_2 that of the centre of gravity of its immersed part. Let moreover the volume of the immersed part be conceived to remain unaltered† whilst the body is in the act of displacement. If each centre of gravity be assumed to ascend, the work of the weight of the body will be represented by $-W.\Delta H_1$, and that of the upward pressure of the fluid by $+W.\Delta H_2$, the negative sign being taken in the former case, because the force acts in a direction opposite to that in which the point of application is moved, and the positive sign in the latter, because it acts in the same direction, so that the aggregate work Σu_2 (see equation 1.) of the forces which constituted the equilibrium of the body in the state from which it has been disturbed is represented by

$$-W.\Delta H_1+W.\Delta H_2.$$

If the centre of gravity of the body or of the displaced fluid *descends* (a property which will be found to characterize a large class of vessels), ΔH_1 in the one case, and ΔH_2 in the other, must be taken with the negative sign, since the weight of the body will be applied in the same direction, and the pressure of the fluid in an opposite direction to that in which their respective points of application are moved. Moreover, the system put in motion includes, with the floating body, the particles of the fluid displaced by it as it changes its position, so that if the weight of any element of the floating body be represented by w_1, and of the fluid by w_2, and if their velocities be v_1 and v_2, the whole *vis viva* is represented by

$$\frac{1}{g}\Sigma w_1 v_1^2+\frac{1}{g}\Sigma w_2 v_2^2,$$

and we have by equation 1,

$$U(\theta)-W(\Delta H_1-\Delta H_2)=\frac{1}{2g}\Sigma w_1 v_1^2+\frac{1}{2g}\Sigma w_2 v_2^2. \quad \ldots \quad (4.)$$

In the extreme position into which the body is made to roll and in which $\Sigma w_1 v_1^2=0$,

$$U(\theta)=W.(\Delta H_1-\Delta H_2)+\frac{1}{2g}\Sigma w_2 v_2^2, \quad \ldots \quad (5.)$$

* When a floating body is so made to incline from any one position into any other as that the volume of fluid displaced by it may in the one position be equal to that in the other, its centre of gravity is also vertically displaced; for if this be not the case, the perpendicular distance of the centre of gravity of the body from its plane of flotation must remain unchanged, and the form of that portion of its surface, which is subject to immersion, must be *determined geometrically* by this condition; but by the supposition the form of the body is undetermined. It is remarkable what currency has been given to the error, that whilst a vessel is rolling or pitching its centre of gravity remains at rest. I should not otherwise have thought this note necessary.

† It will be shown that this supposition is only approximately true.

or if the inertia of the displaced fluid be neglected,

$$U(\theta)=W.(\Delta H_1-\Delta H_2). \quad \quad (6.)$$

Whence it follows that the work necessary to incline a floating body through any given angle is equal to that necessary to raise it bodily through a height equal to the difference of the vertical displacements of its centre of gravity and of that of its immersed part, so that other things being the same, that ship is the most stable the product of whose weight by this difference is the greatest.

In the case in which the centre of gravity of the displaced fluid descends, the *sum* of the displacements is to be taken instead of the difference.

8. This conclusion is nevertheless in error in the following respects:—

1st. It supposes that throughout the motion the weight of the displaced fluid remains equal to that of the floating body, which equality cannot accurately have been preserved by reason of the inertia of the body and of the displaced fluid*.

From this cause there cannot but result small vertical oscillations of the body about those positions which, whilst it is in the act of inclining, correspond to this equality, which oscillations are independent of its principal oscillation.

2ndly. It involves the hypothesis of absolute rigidity in the floating body, so that the motion of every part and its *vis viva* may cease at *once* when the principal oscillation terminates. The frame of a ship and its masts are however elastic, and by reason of this elasticity there cannot but result oscillations, which are independent of, and may not synchronize with, the principal oscillation of the ship as she rolls, so that the *vis viva* of every part cannot be assumed to cease and determine at one and the same instant, as it has been supposed to do.

3rdly. No account has been taken of the work expended in communicating *motion* to the displaced fluid, measured by half its *vis viva* and represented by the term $\frac{1}{2g}\Sigma w_2 v_2^2$ in equation 5.

9. From a careful consideration of these causes of error, I was led to conclude that they would not affect that practical application of the formula which I had principally in view in investigating it, especially as in certain respects they tended to neutralize one another. The question appeared however of sufficient importance to be subjected to the test of experiment, and on my application, the Lords Commissioners of the Admiralty were pleased to direct that such experiments should be made in Her Majesty's Dockyard at Portsmouth, and Mr. FINCHAM, the eminent

* The motion of the centre of gravity of the body being the same as though all the disturbing forces were applied directly to it, it follows, that no elevation of this point is caused in the beginning of the motion, by the application of a horizontal disturbing force, or by a horizontal displacement of the weight of the body, which, if it be a ship, may be effected by moving its ballast. The motion of rotation thereby produced takes place therefore, in the first instance, about the centre of gravity, but it cannot so take place without destroying the equality of the weight of the displaced fluid to that of the body. From this inequality there results a vertical motion of the centre of gravity, and another axis of rotation.

Master Shipwright of that dockyard, and Mr. Rawson were kind enough to undertake them.

The details of these experiments accompany my paper; they extend beyond the object originally contemplated by me; and whether regard be had to the practical importance of the question under discussion, the great care and labour bestowed upon them, or the many expedients by which these gentlemen have succeeded in giving to them a degree of accuracy hitherto, I believe, unknown in experiments of this kind, they claim to rank as authentic and valuable contributions to the science of naval construction.

10. That it might be determined experimentally whether the work which must be done upon a floating body to incline it through a given angle be that represented by equation 6, it was necessary to do upon such a body an amount of work which could be measured; and it was further necessary to ascertain what were the elevations of the centres of gravity of the body and of its immersed part thus produced, and then to see whether the amonnt of work done upon the body equaled the difference of these elevations multiplied by its weight.

To effect this, I proposed that a vessel should be constructed of a simple geometrical form, such that the place of the centre of gravity of its immersed part might readily be determined in every position into which it might be inclined, that of its plane of flotation being supposed to be known; and that a mast should be fixed to it, and a long yard to this mast, and that when the body floated in a vertical position a weight suspended from one extremity of the yard should suddenly be allowed to act upon it causing it to roll over; that the position into which it thus rolled should be ascertained, together with the corresponding elevations of its centre of gravity and the centre of gravity of its immersed part, and the vertical descent of the weight suspended from the extremity of its arm. The product of this vertical descent by the weight suspended from the arm ought then, by the formula, to be found nearly equal to the difference of the elevations of the two centres of gravity multiplied by the weight of the body; and this was the test to which I proposed that the formula should be subjected, having in view its adoption by practical men as a principle of naval construction.

To give to the deflecting weight that *instantaneous* action on the extremity of the arm which was necessary to the accuracy of the experiment, a string was in the first place to be affixed to it and attached to a point vertically above, in the ceiling. When the deflecting weight was first applied this string would sustain its pressure, but this might be thrown at once upon the extremity of the arm by cutting it. A transverse section of the vessel, with its mast and arm, was to be plotted on a large scale on a board, and the extreme position into which the vessel rolled being by some means observed, the water-line corresponding to this position was to be drawn. The position of the yard, in respect to the surface of the water in that position, would then be known, and the vertical descent of the deflecting weight could be measured, and also the vertical ascent of the centre of gravity of the immersed part or displacement.

Fig. 1.

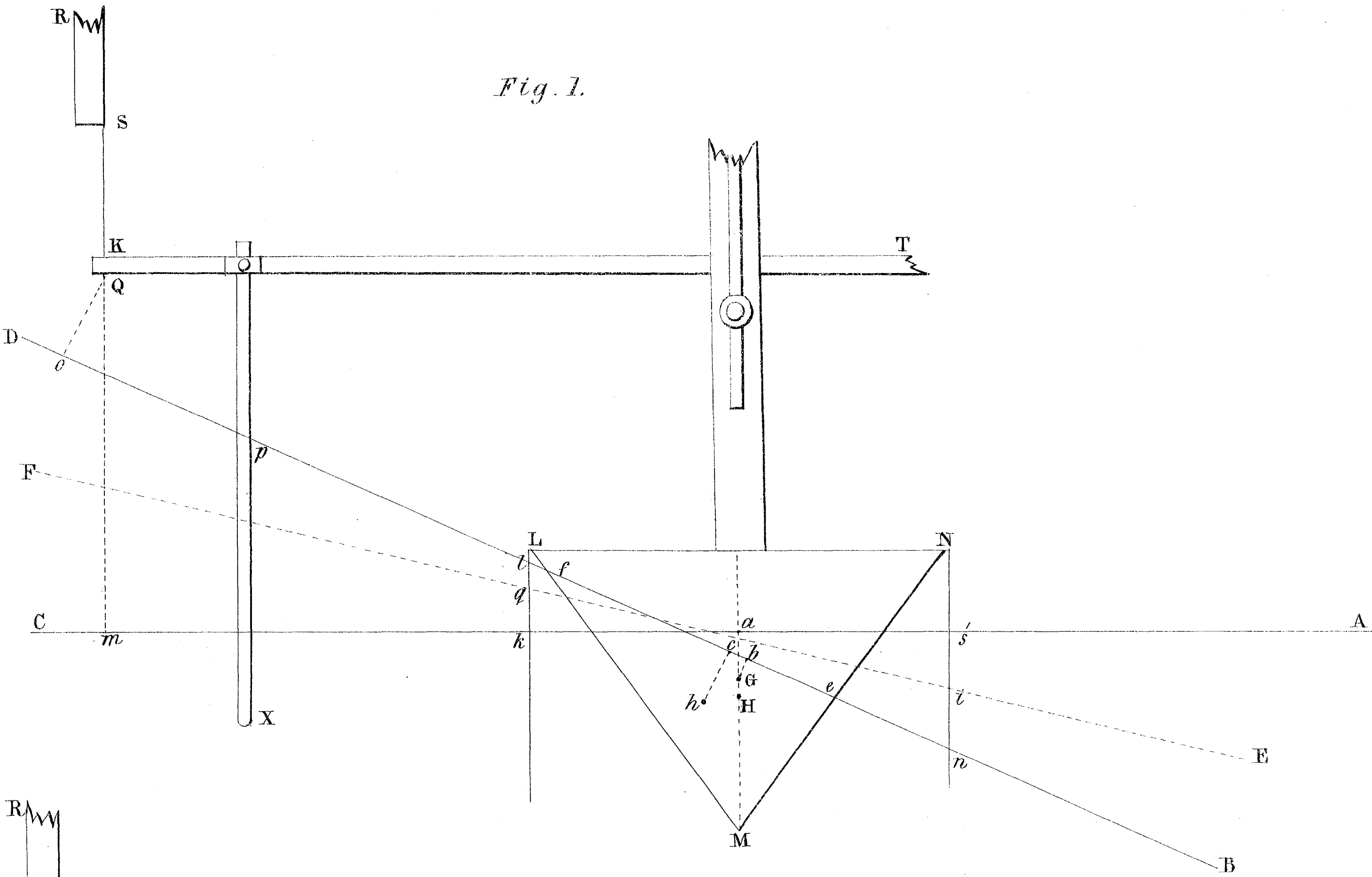
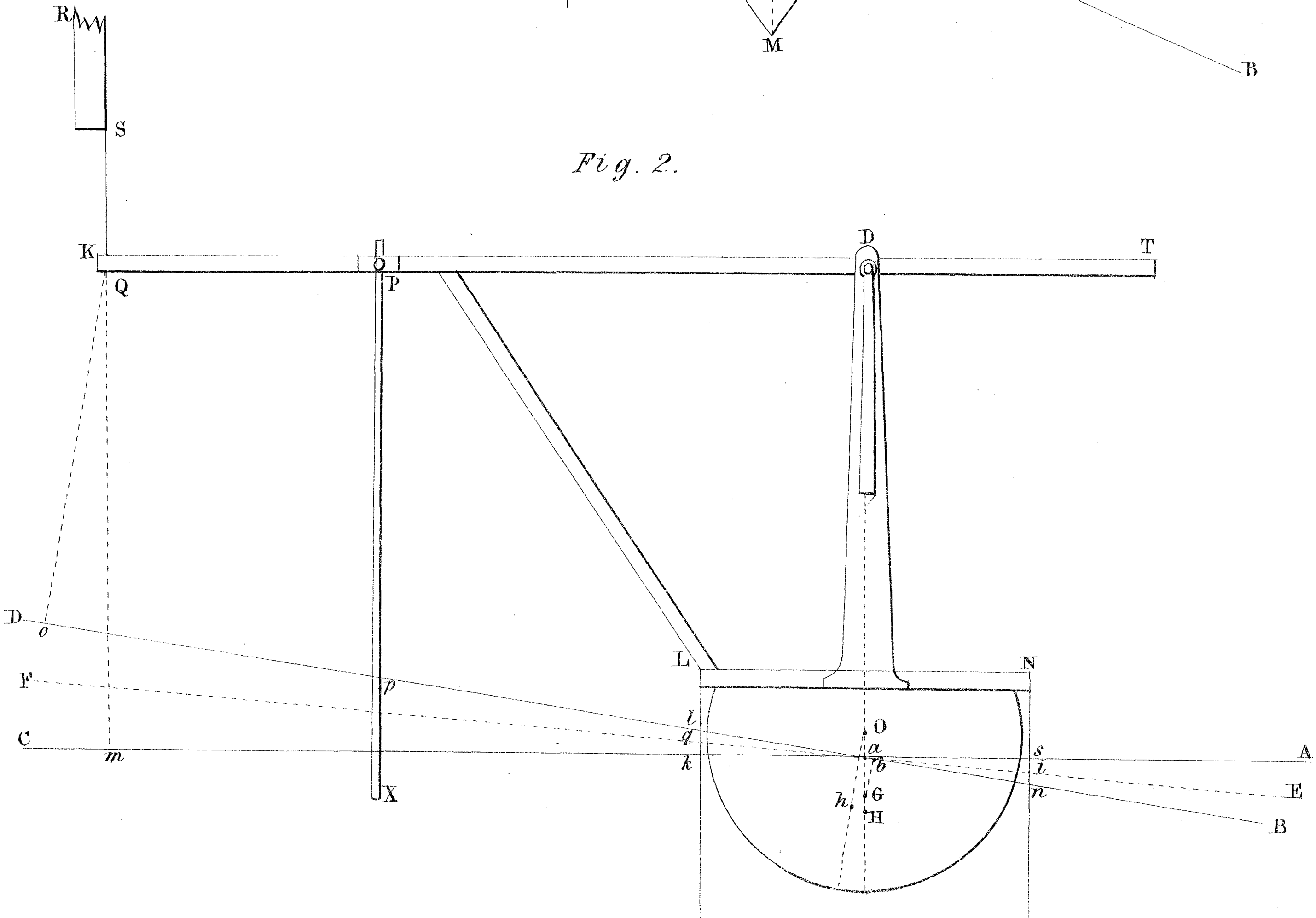


Fig. 2.



J. Basire, Lith.

To determine the position of the centre of gravity of the vessel, it was to be allowed to rest in an inclined position under the action of the deflecting weight; and the water-line corresponding to this position being drawn on the board, the corresponding position of the deflecting weight and of the centre of gravity of the immersion were thence to be determined. The determination of the position of the vertical passing through the centre of gravity of the body would thus become an elementary question of statics; and the intersection of this line, with that about which the section was symmetrical, would mark the position of the centre of gravity, This determination might be verified by a second similar experiment with a different deflecting weight.

These suggestions have received a great development at the hands of Mr. RAWSON, and he has adopted many new and ingenious expedients in carrying them out. Among these, that by which the position of the water-line was determined in the extreme position into which the vessel rolls, appears to me specially worthy of observation. A strip of wood was fastened at right angles to that extremity of the yard to which the deflecting weight was attached, of sufficient length to dip into the water when the vessel rolled, on this slip of wood, and also on the side of the vessel nearest to it, a strip of glazed paper was fixed. The highest points at which these strips of paper were wetted in the rolling of the vessel, were obviously points in the water-line in its extreme position, and being plotted upon the board, a line drawn through them determined that position with a degree of accuracy which left nothing to be desired.

11. Two forms of vessels were used (see Plate XLVII. figs. 1 and 2); one of them had a triangular and the other a semicircular section. The following Table contains the general results of the experiments, of which the particulars are detailed in the Appendix:—

Form of the model experimented on.	No. of experiment.	Weight of model and loading.	Disturbing weight.	Dynamical stability, as determined by experiment.	Dynamical stability calculated from the formula $U(\theta)=W\Delta(H_1-H_2)$.	Extreme inclination into which the vessel rolled, as determined by experiment.	Extreme inclination into which the vessel should roll, as determined by calculation from the formula $U(\theta)=W\Delta(H_1-H_2)$*.	Inclination in which the vessel finally rested when subjected to the action of the disturbing weight.	Ratio of the volume of the displaced fluid in the extreme position into which the vessel rolled to that in the position in which it originally rested.
		lbs.	lbs.			° ′	° ′	° ′	
Triangular model.	1.	33·8626	·5485	·5161	·5361	23 30	....	12 30	·8961
	2.	36·8590	·3450	·4887	·4951	15 30	....	8 0	·98114
	3.	37·3563	·5377	1·1724	1·4503	24 0	....	13 0	·88512
	4.	38·2911	·5739	1·2673	1·8460	25 0	....	13 30	·9330
Circular model.	1.	197·18	2·8225	7·3761	7·394	26 0	24 20	13 0	
	2.	197·18	1·9570	3·2486	3·122	17 0	16 22	9 0	
	3.	255·43	1·9570	1·7727	1·7667	10 0	10 0	4 30	

In the experiments with the smaller triangular model the differences between the

* The inclinations are calculated by the formula (9.).

results and those given by the formula are much greater than in the experiments with the heavier cylindrical vessel.

In explanation of this difference, it will be observed, *first*, that the conditions of the experiment with the cylindrical model more nearly approach to those which are assumed in the formula than those with the other; the disturbance of the water in the change of the position of the former being less, and therefore the work expended upon the inertia of the water, of which the formula takes no account, less in the one case than the other; and, *secondly*, that the weight of the model being greater, this inertia bears a less proportion to the amount of work required for inclining it than in the other case.

The effect of this inertia adding itself to the buoyancy of the fluid, cannot but be to lift the vessel out of the water and to cause the displacement to be less at the termination of each rolling oscillation than at its commencement*. This variation in volume of the displacement was apparent in all the experiments. Its amount was measured and is recorded in the last column of the Table; its tendency is to produce in the body vertical oscillations, which are so far independent of its rolling motion that they will not probably synchronize with it. The body displacing, when rolling, less fluid than it would at rest, the effect of the weight used in the experiments to incline it is thereby increased, and thus is explained the fact (apparent in the eighth and ninth columns of the Table) that the inclination by experiment is somewhat greater than the formula would make it.

12. *The dynamical stability of a vessel whose athwart sections (where they are subject to immersion and emersion) are circular, having their centres in a common axis.*

Let EDF, Plate XLVIII. fig. 3 or 4, be an athwart section of such a vessel, the parts of whose periphery ES and FR, subject to immersion and emersion, are parts of the same circular arc ETF, whose centre is C. Let G_1 represent the projection of the centre of gravity of the vessel on this section, and G_2 that of the centre of gravity of the space whose section is SDRT, supposing it filled with water. This space lies wholly within the vessel in fig. 3 and without it in fig. 4. Let

$h_1 = CG_1$, $h_2 = CG_2$.

W_1=weight of vessel.

W_2=weight of water occupying, or which would occupy, the space whose section is STRD.

θ =the inclination from the vertical.

Since in the act of the inclination of the vessel the whole volume of the displaced fluid remains constant, and also that volume of which STRD is the section†, it follows that the volume of that portion of which the circular area PSRQ is the section

* This result connects itself with the well-known fact of the rise of a vessel out of the water when propelled rapidly, which is so great in the case of fast track-boats, as considerably to reduce the resistance upon them.

† It will be observed that the space STRD is supposed always to be under water.

Fig. 4.

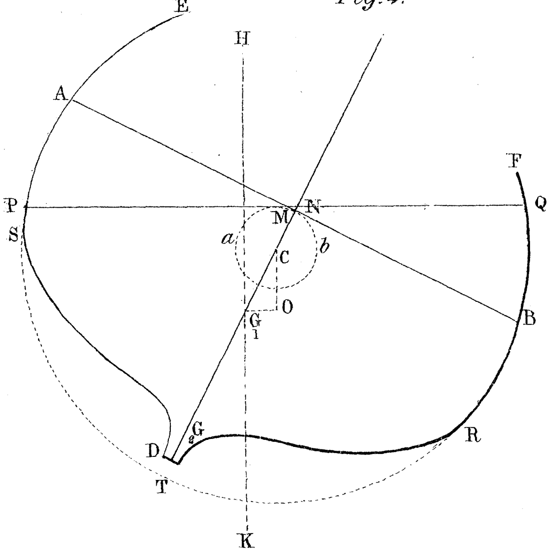


Fig. 3.

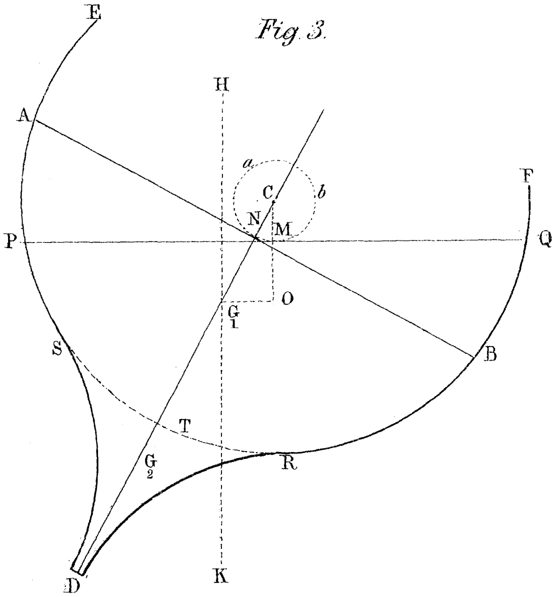


Fig. 5.

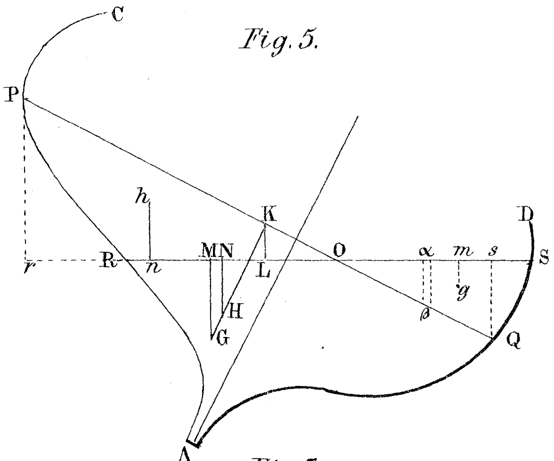


Fig. 6.

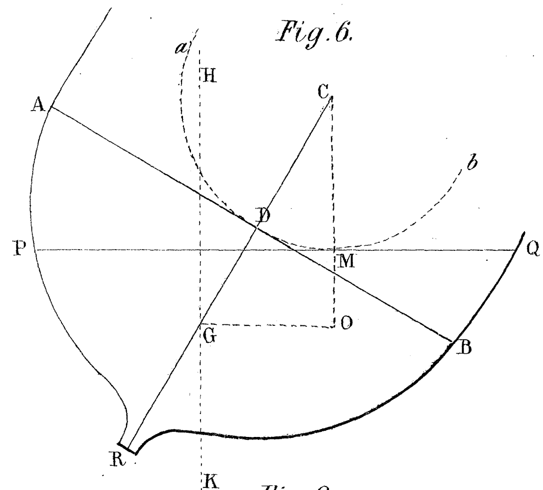


Fig. 7.

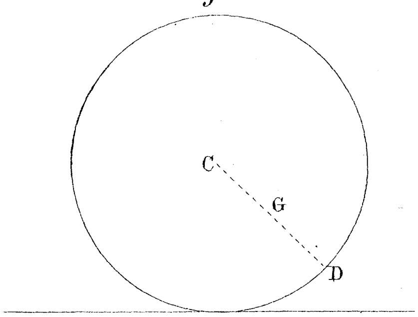


Fig. 9.

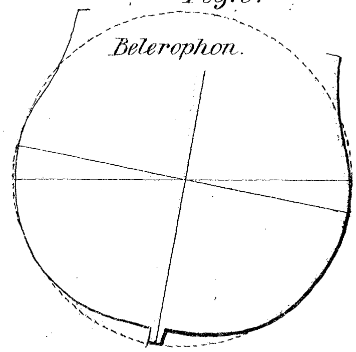


Fig. 8.

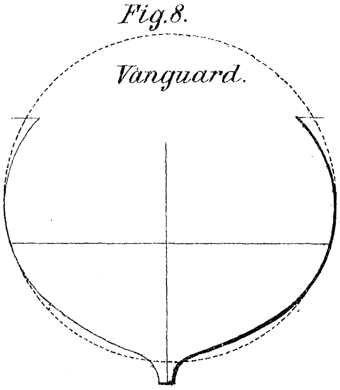
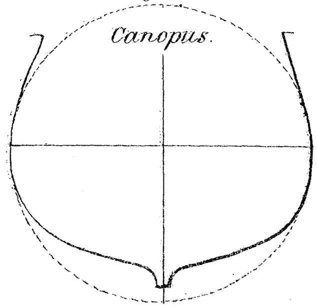


Fig. 10.



remains also constant, and that the water-line PQ, which is the chord of that area, remains at the same distance from C, so that the point C neither ascends nor descends. Now the forces which constituted the equilibrium of the vessel in its vertical position were its weight and that of the fluid it displaced. Since the point C is not vertically displaced, the work of the former force, as the body inclines through the angle θ, is represented by $-W_1h_1$ vers θ. The work of the latter is equal to that of the upward pressure of the water which would occupy the space of which the circular area PTQ is the section *increased*, in the case represented in fig. 3, by that of the water which would occupy STRD; and *diminished* by it in the case represented in fig. 4.

But since the space, of which the circular area PTQ is the section, remains similar and equal to itself, its centre of gravity remains always at the same distance from the centre C, and therefore neither ascends nor descends. Whence it follows that the work of the water which would occupy this space is *zero*; so that the work of the *whole* displaced fluid is equal to that of the *part* of it which occupies the space STRD, taken in the case represented in fig. 3 with the positive, and in that represented in fig. 4 with the negative sign. It is represented therefore generally by the formula $\pm W_2h_2$ vers θ. On the whole, therefore, the work Σu_2 (see Art. 1.) of those forces, which in the vertical position of the body constituted its equilibrium, is represented by the formula

$$\Sigma u_2 = -W_1h_1 \text{ vers } \theta \pm W_2h_2 \text{ vers } \theta.$$

Representing therefore the dynamical stability Σu_1 by $U(\theta)$, we have by equation (2.)

$$U(\theta) = (W_1h_1 \mp W_2h_2) \text{ vers } \theta, \quad \ldots \ldots \quad (7.)$$

in which expression the sign $\mp$ is to be taken according as the circular area ATB lies wholly within the area ADB, as in fig. 3, or partly without it, as in fig. 4. Other things being the same, the latter is therefore a more stable form than the former.

13. The work of the upward pressure of the water upon the vessel represented in fig. 4 being a negative quantity, $-W_2h_2$ vers θ, it follows that the point of application of the pressure must be moved in a direction opposite to that in which the pressure acts; but the pressure acts upwards, therefore its point of application, *i. e.* the centre of gravity of the displaced fluid, *descends*. This property may be considered to distinguish *mechanically* the class of vessels whose type is fig. 3, from that class whose type is fig. 4; as the property of including wholly or only partly, within the area of any of their athwart sections, the corresponding circular area ETF, distinguishes them geometrically.

14. To obtain from the formula 7 an expression adapted to the experiments with the circular model, Plate XLVII. fig. 2, let

$$OM = b, \; MQ = c, \; \text{Disturbing weight} = w.$$

Now it may readily be shown that the vertical descent of the point Q, when the vessel is made to incline through the angle θ, is represented by

$$b \text{ vers } \theta + c \sin \theta.$$

Therefore the work done upon the vessel by the disturbing weight in the act of this inclination is represented by

$$w(b \text{ vers } \theta + c \sin \theta),$$

which expression ought, therefore, neglecting the inertia of the fluid, to equal that (equation 7.) which represents $U(\theta)$, or

$$w(b \text{ vers } \theta + c \sin \theta) = (W_1 h_1 \mp W_2 h_2) \text{ vers } \theta,$$

whence we obtain

$$\tan \tfrac{1}{2}\theta = \frac{wc}{W_1 h_1 \mp W_2 h_2 - wb} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

In the vessel experimented upon, $W_2 = 0$,

$$\therefore \tan \frac{1}{2}\theta = \frac{wc}{W_1 h_1 - wb}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

which is the formula used in calculating the eighth column of the Table, p. 615.

15. *The dynamical stability of a vessel of any given form subjected to a rolling or pitching motion.*

Conceive the vessel, after having completed an oscillation in any given direction,—being then about to return towards its vertical position—to be for an instant at rest, and let RS (fig. 5) represent the intersection of its plane of flotation *then*, and PQ of its flotation when in its vertical position, with a section CAD of the vessel perpendicular to the mutual intersection O of these planes. The section CAD will then be a vertical section of the vessel.

Let G be the projection upon it of the vessel's centre of gravity when in its vertical position.

H, that of the centre of gravity of the fluid displaced by the vessel in the vertical position.

g, that of the fluid displaced by the portion of the vessel of which QOS is a section.

h, that of the fluid which would be displaced by the portion, of which POR is a section, if it were immersed.

GM, HN, gm, hn, KL perpendiculars upon the plane RS.

W=weight of vessel or of displaced fluid.

w=weight of water displaced by either of the equal portions of the vessel of which POR and QOS are sections.

H_1=depth of centre of gravity of vessel in vertical position.

H_2=depth of centre of gravity of displaced water in vertical position.

ΔH_1=elevation of centre of gravity of vessel.

ΔH_2=elevation of centre of gravity of displaced water.

θ =inclination of planes PQ and RS.

η =inclination of line O in which planes PQ and RS intersect, to that line about which the plane PQ is symmetrical.

ζ = inclination to horizon of line about which the plane PQ is symmetrical.

x = distance of section CAD, measured along the line whose projection is O, from the point where that line intersects the midship section.

$y = O\beta$.

$y_1 = PQ$.

$y_2 = RS$.

$z = hn + mg$.

$\lambda = KL$.

I = moment of inertia of plane PQ about axis O.

A and B = moments of inertia of PQ about its principal axes.

μ = weight of a cubic unit of water.

Suppose the water actually *displaced* by the vessel to be, on the contrary, *contained* by it; and conceive that which occupies the space QOS to pass into the space POR, the whole becoming solid. Let ΔH_3 represent the corresponding elevation of the centre of gravity of the whole contained fluid. Then will $\Delta H_2 + \Delta H_3$ represent the total elevation of the centre of gravity of this fluid as it passes from the position it occupied when the vessel was vertical into the position PAQ. But this elevation is obviously the same as though the fluid had assumed the solid state in the vertical position of the body, and the latter had revolved with it, in that state, into its present position. It is therefore represented by KH−NH*;

$$\therefore \Delta H_2 + \Delta H_3 = KH - NH \text{ and } \Delta H_3 = KH - NH - \Delta H_2.$$

Since, moreover, by the elevation of the fluid in QOS, whose weight is w, into the space OPR, and of its centre of gravity through $(gm+hn)$, the centre of gravity of mass of fluid of which it forms a part, and whose weight is W, is raised through the space ΔH_3; it follows, by a well-known property of the centre of gravity of a system†, that

$$W.\Delta H_3 = w(gm+hn);$$

$$\therefore W(KH - NH - \Delta H_2) = w(gm+hn).$$

But

$$NH = KH\cos\theta - KL = H_2\cos\theta - \lambda;$$

$$\therefore KH - NH = H_2 \text{ vers } \theta + \lambda,$$

and

$$mg + nh = z;$$

$$\therefore W(H_2 \text{ vers } \theta + \lambda - \Delta H_2) = wz;$$

$$\therefore W.\Delta H_2 = W(H_2 \text{ vers } \theta + \lambda) - wz \quad \quad (10.)$$

* The line joining the centres of gravity of the vessel and its immersed part, in its vertical position, is parallel to the plane CAD, for it is perpendicular to the plane PQ, to whose intersection with the plane RS the plane CAD is perpendicular; $\therefore$ GK=H_1 and HK=H_2.

† Poncelet, Mécanique Industrielle, 2[me] partie, Art. 50, or Moseley's Mechanical Principles of Engineering, Art. 59.

Also $\Delta H_1 = KG - MG = H_1 - (H_1 \cos\theta - \lambda) = H_1 \text{ vers } \theta + \lambda$;

$\therefore W(\Delta H_1 \mp \Delta H_2) = W(H_1 \mp H_2) \text{ vers } \theta + wz$* ;

$\therefore$ (equation 6.) $U(\theta, \eta) = W(H_1 \mp H_1) \text{ vers } \theta + wz$; . . . . (11.)

the sign $\mp$ being taken according as the vessel is of the class represented in fig. 3, in which the centre of gravity of the displaced fluid ascends, or of that represented in fig. 4, in which it descends.

If $\alpha\beta$ be a vertical prismatic element of the space QOS, whose base is $dx\,dy \cos\theta$, and height $y \sin\theta$, then will $w.\overline{mg}$ be represented, in respect to that element, by $\mu y \sin\theta . dx\,dy \cos\theta . \frac{1}{2} y \sin\theta$, or by $\frac{1}{2}\mu \sin^2\theta \cos\theta\, y^2 dx\,dy$; and wz will be represented, in respect to the whole space of which $\text{P}r s\text{Q}$ is the section, by

$$\frac{1}{2}\mu \sin^2\theta \cos\theta \iint y^2 dx\,dy,$$

or by
$$\frac{1}{2}\mu \sin^2\theta \cos\theta . \text{I}.$$

If therefore we represent by φ the value of wz, in respect to the spaces of which the mixtilinear areas P Rr and QSs are the sections, we have

$$wz = \frac{1}{2}\mu \text{I} \sin^2\theta \cos\theta + \varphi.$$

But the axis O, about which the moment of inertia of the plane PQ is I, is inclined to the principal axes of that plane at the angles η and $\frac{\pi}{2} - \eta$, about which principal axes the moments of inertia are A and B; and it has been shown by M. DUPIN† that when θ is small the line in which the planes PQ or RS intersect passes through the centre of gravity of each;

$$\therefore \quad \text{I} = \text{A} \cos^2\eta + \text{B} \sin^2\eta;$$

therefore by equation (11.),

$$U(\theta, \eta) = W(H_1 \mp H_2) \text{ vers } \theta + \frac{1}{2}\mu(\text{A} \cos^2\eta + \text{B} \sin^2\eta) \sin^2\theta \cos\theta + \varphi. \quad . \ . \ . \ . \ (12.)$$

If θ be so small that the spaces PrR and QsS are evanescent in comparison with POr and QOs, then, assuming $\varphi = 0$ and $\cos\theta = 1$,

$$U(\theta, \eta) = W(H_1 \mp H_2) \text{ vers } \theta + \frac{1}{2}\mu(\text{A} \cos^2\eta + \text{B} \sin^2\eta) \sin^2\theta, \quad . \ . \ . \ . \ . \ . \ (13.)$$

which may be put under the form

$$U(\theta, \eta) = \left\{ W(H_1 \mp H_2) + \mu(\text{A} \cos^2\eta + \text{B} \sin^2\eta) \right\} \text{vers } \theta.$$

Again, since

$$\sin\zeta = \sin\theta \sin\eta, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (14.)$$

* The sign $\mp$ is here taken to include the case in which the centre of gravity of the displaced fluid descends. See Art. 7.

† Sur la Stabilité des Corps Flottants, p. 32.

and $(A\cos^2\eta+B\sin^2\eta)\sin^2\theta=\{A+(B-A)\sin^2\eta\}\sin^2\theta$,

$\therefore$ $(A\cos^2\eta+B\sin^2\eta)\sin^2\theta=A\sin^2\theta+(B-A)\sin^2\zeta$;

$\therefore$ by equation 13,

$$U(\theta,\zeta)=W(H_1\mp H_2)\text{ vers }\theta+\frac{1}{2}\mu\{A\sin^2\theta+(B-A)\sin^2\zeta\}, \quad . \quad . \quad . \quad . \quad (15.)$$

by which formula the dynamical stability of the ship is represented, both as it regards a pitching and a rolling motion.

If in equation 13 $\eta=\frac{\pi}{2}$, the line in which the plane PQ (parallel to the deck of the ship) intersects its plane of flotation is at right angles to the length of the ship, and we have, since in this case $\theta=\zeta$ (see equation 14.),

$$U(\zeta)=W(H_1\mp H_2)\text{ vers }\zeta+\frac{1}{2}\mu B\sin^2\zeta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

which expression represents the dynamical stability, in regard to a pitching motion alone, as the equation

$$*U(\theta)=W(H_1\mp H_2)\text{ vers }\theta+\frac{1}{2}\mu A\sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

represents it in regard to a rolling motion alone.

16. If a *given* quantity of work represented by $U(\theta)$ be supposed to be done upon the vessel, the angle θ through which it is thus made to roll may be determined by solving equation 17 with respect to $\sin\frac{\theta}{2}$. We thus obtain

$$\sin^2\frac{\theta}{2}=\frac{W(H_1\mp H_2)+\mu A-\sqrt{\{W(H_1\mp H_2)+\mu A\}^2-2\mu A.U(\theta)}}{2\mu A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

17. If PR and QS be conceived to be straight lines, so that POR and QOS are triangles, then $w.z$, taken in respect to an element included between the section CAD, and another parallel to it and distant by the small space dx, is represented by

$$\frac{1}{4}\mu y_1 y_2\sin\theta dx(mg+nh);$$

or, since $$mg+nh=\frac{1}{3}y_1\sin\theta,$$

by $$\frac{1}{12}\mu\sin^2\theta y_1^2 y_2 dx;$$

$$\therefore \quad wz=\frac{1}{12}\mu\sin^2\theta\int y_1^2 y_2 dx,$$

and, equation 11

$$U(\theta,\zeta)=W(H_1\mp H_2)\text{ vers }\theta+\frac{1}{12}\mu\sin^2\theta\int y_1^2 y_2 dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

which formula may be considered an approximate measure of the stability of the vessel under all circumstances.

* This formula may be verified experimentally by a method similar to that applied to equation 6. See Art. 10.

If, as in the case of the experiments of Messrs. FINCHAM and RAWSON, the vessel be prismatic and the direction of the disturbance perpendicular to its axis,

$$y=\text{constant}=a, \text{ and } z=\frac{1}{3}a\sin\theta;$$

$$\therefore\quad wz=\frac{1}{3}aw\sin\theta, \text{ and}$$

$$U(\theta)=W(H_1\mp H_2)\text{ vers }\theta+\frac{1}{3}aw\sin\theta.$$

Mr. RAWSON has obligingly undertaken the verification of this formula by comparing it with his experiments on the cylindrical model. The following is the result:—

No. of experiment.	W.	w.	H_1.	H_2.	θ.	$U(\theta)$ by formula.	$U(\theta)$ by experiment.
	lbs.	lbs.			°		
3	255·43	17·294	3·903	4·800	9	1·760	1·766
4	255·43	46·84	4·02	4·82	26	13·478	13·5015
5	197·18	37·98	0·80	3·80	26	6·807	7·3761

18. *A rigid surface on which the vessel may be supposed to rest whilst in the act of pitching and rolling.*

If we imagine the position of the centre of gravity of a vessel afloat to be continually changed by altering the positions of some of its contained weights without altering the weight of the whole, so as to cause the vessel to incline into an infinite number of different positions displacing, in each, the same volume of water, then will the different planes of flotation, corresponding to these different positions, envelope a curved surface, called the surface of the planes of flotation (*surface des flotaisons*), whose properties have been discussed at length by M. DUPIN in his excellent memoir, Sur la Stabilité des Corps Flottants, which forms part of his Applications de Géométrie*. So far as the properties of this surface concern the conditions of the vessel's *equilibrium*, they have been exhausted in that memoir, but the following property, which has reference rather to the conditions of its dynamical stability than its equilibrium, is not stated by M. DUPIN:—

If we conceive the surface of the planes of flotation to become a rigid surface, and also the surface of the fluid to become a rigid plane without friction, so that the former surface may rest upon the latter and roll and slide upon it, the other parts of the vessel being imagined to be so far immaterial as not to interfere with this motion, but not so as to take away their weight or to interfere with the application of the upward pressure of the fluid to them, then will the motion of the vessel, when resting by this curved surface upon this rigid but perfectly smooth horizontal plane, be the same as it was when, acted upon by the same forces, it rolled and pitched in the fluid.

In this general case of the motion of a body resting by a curved surface upon a horizontal plane, that motion may be, and generally will be, of a complicated cha-

* BACHELIER, Paris, 1822.

racter, including a sliding motion upon the plane, and simultaneous motions round two axes passing through the point of contact of the surface with the planes and corresponding with the rolling and pitching motion of a ship. It being however possible to determine these motions by the known laws of dynamics, when the form of the surface of the planes of flotation is known, the complete solution of the question is involved in the determination of the latter surface.

The following property*, proved by M. Dupin in the memoir before referred to (p. 32), effects this determination :—

"The intersection of any two planes of flotation, infinitely near to each other, passes through the centre of gravity of the area intercepted upon either of these planes by the external surface of the vessel."

If, therefore, any plane of flotation be taken, and the centre of gravity of the area here spoken of be determined with reference to that plane of flotation, then that point will be one in the curved surface in question, called the surface of the planes of flotation, and by this means any number of such points may be found and the surface determined.

19. *The axis about which a vessel rolls may be determined, the direction in which it is rolling being given.*

If, after the vessel has been inclined through any angle, it be left to itself, the only forces acting upon it (the inertia of the fluid being neglected) will be its weight and the upward pressure of the fluid it displaces; the motion of its centre of gravity will therefore, by a well-known principle of mechanics, be wholly in the same vertical line.

Let HK (fig. 6) represent this vertical line, PQ the surface of the fluid, and aMb the surface of the planes of flotation. As the centre of gravity G traverses the vertical HK, this surface will partly roll and partly slide by its point of contact M on the plane PQ.

If we suppose, therefore, PRQ to be a section of the vessel through the point M, and perpendicular to the axis about which it is rolling, and if we draw a vertical line MO through the point M, and through G a horizontal line GO parallel to the plane PRQ, then the position of the axis will be determined by a line perpendicular to these, whose projection on the plane PRQ is O.

For since the motion of the point G is in the vertical line HK, the axis about which the body is revolving passes through GO, which is perpendicular to HK; and since the point M of the vessel traverses the line PQ, the axis passes also through MO which is perpendicular to PQ; and GO is drawn parallel to, and MO *in* the plane PRQ, which, by supposition, is perpendicular to the axis, therefore the axis is perpendicular to GO and MO.

If HK be in the plane PRQ, which is the case whenever the motion is exclusively one of rolling or one of pitching, the point O is determined by the intersection of GO and MO.

20. *The time of the rolling through a small angle of a vessel whose athwart sections*

* This property appears to have been first given by Euler.

are (in respect to the parts subject to immersion and emersion) circular, and have their centres in the same longitudinal axis.

Let EDF (fig. 3 or fig. 4) represent the midship section of such a vessel, in which section let the centre of gravity G_1 be supposed to be situated, and let HK be the vertical line traversed by G_1 as the vessel rolls. Imagine it to have been inclined from its vertical position through a given angle θ_1 and the forces which so inclined it then to have ceased to act upon it, so as to have allowed it to roll freely back again towards its position of equilibrium until it had attained the inclination OCD to the vertical, which suppose to be represented by θ.

Referring to equation 1., let it be observed that in this case $\Sigma u_2=0$, so that the motion is determined by the condition

$$\Sigma u_1=\frac{1}{2g}\Sigma wv^2. \quad \ldots \ldots \quad (20.)$$

But the forces which have displaced it from the position in which it was, for an instant, at rest are its weight and the upward pressure of the water; and the work of these, $U(\theta_1)-U(\theta)$, done between the inclinations θ and θ_1 when the vessel was in the act of receding from the vertical, was shown to be represented by $(W_1h_1\mp W_2h_2)$ $(\text{vers}\,\theta-\text{vers}\,\theta_1)$ by Art. 12 (adopting the same notation as in that article); therefore the work, between the same inclinations, when the motion is in the opposite direction, is represented by the same expression with the sign changed;

$$\therefore \quad \Sigma u_1=(W_1h_1\mp W_2h_2)(\text{vers}\,\theta_1-\text{vers}\,\theta),$$

and since the axis about which the vessel is revolving is perpendicular to the plane EDF, and passes through the point O (Art. 19.), if W_1k_2 represent its moment of inertia about an axis perpendicular to the plane EDF, and passing through its centre of gravity G_1,

$$\Sigma wv^2=W_1(k^2+\overline{OG}_1^2)\left(\frac{d\theta}{dt}\right)^2.$$

Substituting in equation 20 and writing for OG_1 its value $h_1\sin\theta$, we have

$$(W_1h_1\mp W_2h_2)(\text{vers}\,\theta_1-\text{vers}\,\theta)=\frac{W_1}{2g}(k^2+h_1^2\sin^2\theta)\left(\frac{d\theta}{dt}\right)^2;$$

$$\therefore \; t(\theta_1)=\frac{-1}{\sqrt{2gh_1\left(1\mp\frac{W_2h_2}{W_1h_1}\right)}}\int_{+\theta_1}^{-\theta_1}\sqrt{\frac{k^2+h_1^2\sin^2\theta}{\text{vers}\theta_1-\text{vers}\,\theta}}\,d\theta. \quad \ldots \ldots \quad (21.)$$

$$t(\theta_1)=\frac{1}{\sqrt{2gh_1\left(1\mp\frac{W_2h_2}{W_1h_1}\right)}}\int_{-\theta_1}^{+\theta_1}\sqrt{\frac{k^2+4h_1^2\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta}{2\left(\sin^2\frac{1}{2}\theta_1-\sin^2\frac{1}{2}\theta\right)}}\,d\theta$$

$$=\frac{1}{\sqrt{gh_1\left(1\mp\frac{W_2h_2}{W_1h_1}\right)}}\int_{-\theta_1}^{+\theta_1}\sqrt{\frac{k^2\sec^2\frac{1}{2}\theta+4h_1^2\sin^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta_1-\sin^2\frac{1}{2}\theta}}\cdot\cos\frac{1}{2}\theta\,d\frac{1}{2}\theta,$$

or assuming θ to be so small that the fourth and all higher powers of $\sin\frac{1}{2}\theta$ may be neglected, and observing that, this being the case,

$$\sqrt{k^2\sec^2\frac{1}{2}\theta+4h_1^2\sin^2\frac{1}{2}\theta}=\sqrt{k^2\left(1+\sin^2\frac{1}{2}\theta\right)+4h_1^2\sin^2\frac{1}{2}\theta}$$

$$=k\sqrt{1+\frac{4h_1^2+k^2}{k^2}\sin^2\frac{1}{2}\theta}=k\left\{1+\frac{4h_1^2+k^2}{2k^2}\sin^2\frac{1}{2}\theta\right\}$$

$$t(\theta_1)=\frac{k}{\sqrt{gh_1\left(1\mp\frac{W_2h_2}{W_1h_1}\right)}}\int_{-\theta_1}^{+\theta_1}\frac{1+\frac{4h_1^2+k^2}{2k^2}\sin^2\frac{1}{2}\theta}{\sqrt{\sin^2\frac{1}{2}\theta_1-\sin^2\frac{1}{2}\theta}}d\sin\frac{1}{2}\theta.$$

But

$$\int_{-\theta_1}^{+\theta_1}\frac{d\sin\frac{1}{2}\theta}{\sqrt{\sin^2\frac{1}{2}\theta_1-\sin^2\frac{1}{2}\theta}}=\pi,$$

and

$$\int_{-\theta_1}^{+\theta_1}\frac{\sin^2\frac{1}{2}\theta\, d\sin\frac{1}{2}\theta}{\sqrt{\sin^2\frac{1}{2}\theta_1-\sin^2\frac{1}{2}\theta}}=\frac{1}{2}\pi\sin^2\frac{1}{2}\theta_1,$$

$$\therefore\quad t(\theta_1)=\frac{\pi k}{\sqrt{gh_1\left(1\mp\frac{W_2h_2}{W_1h_1}\right)}}\left\{1+\frac{4h_1^2+k^2}{4k^2}\sin^2\frac{1}{2}\theta_1\right\}. \quad (22.)$$

The sign $\mp$ being taken according as the centre of gravity of the displaced fluid ascends or descends.

21. *The time of a vessel's rolling or pitching through a small angle, its form and dimensions being any whatever.*

Let EDF (figs. 3 or 4) represent the midship section of such a vessel, supposed to be rolling about an axis whose projection is O; and let C represent the centre of the circle of curvature of the surface of its planes of flotation (Art. 18.) at the point M where that surface is touched by the plane PQ, being above the load water-line AB in fig. 3, and beneath it in fig. 4. Let the radius of curvature CM be represented by ϱ; then adopting the same notation as in the last article, and observing that the axis O about which the vessel is turning is perpendicular to EDF, we shall find its moment of inertia to be represented by

$$W_1\{k^2+(H_1\mp\varrho)^2\sin^2\theta\}\left(\frac{d\theta}{dt}\right)^2,$$

where H_1 represents the depth of the centre of gravity in the vertical position of the vessel.

Also, by equation 17, reasoning as in Art. 20,

$$\Sigma u_1=U(\theta_1)-U(\theta)=W_1(H_1\mp H_2)(\cos\theta-\cos\theta_1)+\frac{1}{2}\mu A(\cos^2\theta-\cos^2\theta_1),$$

$\therefore$ by equation 20,

$$W_1(H_1 \mp H_2)(\cos\theta - \cos\theta_1) + \frac{1}{2}\mu A(\cos^2\theta - \cos^2\theta_1) = \frac{W_1}{2g}\left\{k^2 + (H_1 \mp \varrho)^2 \sin^2\theta\right\}\left(\frac{d\theta}{dt}\right)^2$$

$$\therefore\ t(\theta_1) = \frac{1}{\sqrt{2g}}\int_{-\theta_1}^{+\theta_1}\sqrt{\frac{k^2 + (H_1 \mp \varrho)^2 \sin^2\theta}{(H_1 \mp H_2)(\cos\theta - \cos\theta_1) + \frac{1}{2}\frac{\mu A}{W_1}(\cos^2\theta - \cos^2\theta_1)}}\, d\theta$$

$$t(\theta_1) = \frac{1}{\sqrt{2g}}\int_{-\theta_1}^{+\theta_1}\sqrt{\frac{k^2 + (H_1 \mp \varrho)^2 \sin^2\theta}{\left\{H_1 \mp H_2 + \frac{1}{2}\frac{\mu A}{W_1}(\cos\theta + \cos\theta_1)\right\}\left\{\cos\theta - \cos\theta_1\right\}}}\, d\theta.$$

Assuming θ and θ_1 to be so small that $\cos\theta + \cos\theta_1 = 2$, and observing that

$$\cos\theta - \cos\theta_1 = \text{vers}\,\theta_1 - \text{vers}\,\theta,$$

$$t(\theta_1) = \frac{1}{\sqrt{2g\left\{H_1 \mp H_2 + \frac{\mu A}{W_1}\right\}}}\int_{-\theta_1}^{+\theta_1}\sqrt{\frac{k^2 + (H_1 \mp \varrho)^2 \sin^2\theta}{\text{vers}\,\theta_1 - \text{vers}\,\theta}}\,.\,d\theta.$$

Supposing, moreover, ϱ to remain constant between the limits $-\theta_1$ and $+\theta_1$, and integrating as in Art. 20, equation 21,

$$t(\theta_1) = \frac{\pi k}{\sqrt{g\left(H_1 \mp H_2 + \frac{\mu A}{W_1}\right)}}\left\{1 + \frac{4(H_1 \mp \varrho)^2 + k^2}{4k^2}\sin^2\frac{1}{2}\theta_1\right\} \quad \ldots \quad (23.)$$

where ϱ is to be taken with the sign $\mp$ according as the surface of the planes of flotation is above or below the load-water line, and H_2, according as the centre of gravity of the displaced fluid ascends or descends.

Since the value of $\sin^2\frac{1}{2}\theta_1$ is exceedingly small, the oscillations are nearly tautochronous, and the period of each is nearly represented by the formula

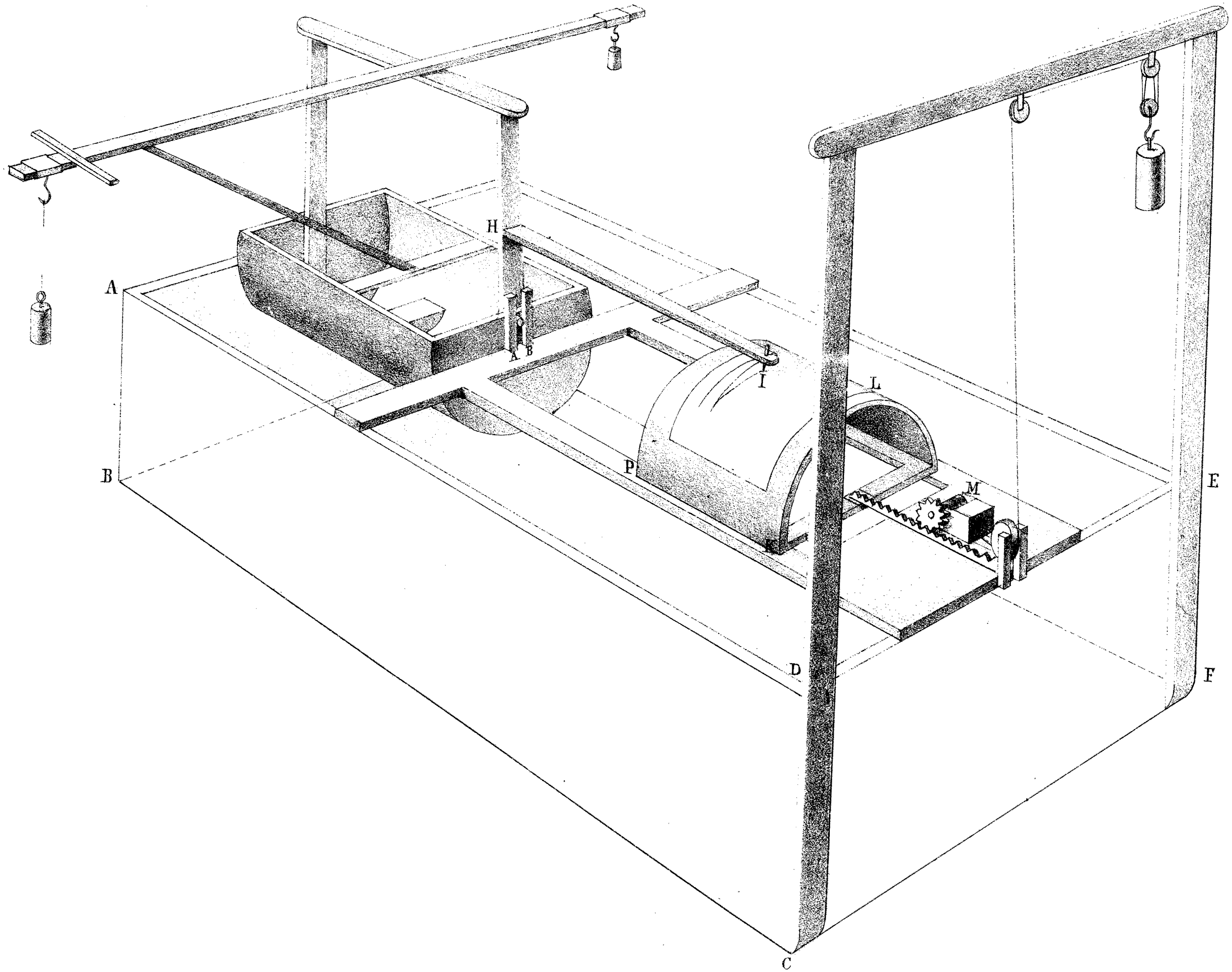
$$t(\theta_1) = \frac{\pi k}{\sqrt{g\left(H_1 \mp H_2 + \frac{\mu A}{W_1}\right)}} \quad \ldots \quad (24.)$$

The following method is given by M. Dupin for determining the value of ϱ*:—

"If the periphery of the plane of flotation be imagined to be loaded at every point with a weight represented by the tangent of the inclination of the sides of the vessel at that point to the vertical, then will the moments of inertia of that curve, so loaded, about its two principal axes, when divided by the area of the plane of flotation, represent the radii of greatest and least curvature of the envelope of the planes of flotation."

If ϱ be taken to represent the radius of greatest curvature, the formula 25 will represent the time of the vessel's rolling; if the radius of least curvature (B being also substituted for A), it will represent the time of pitching.

* Applications de Géométrie, p. 47.

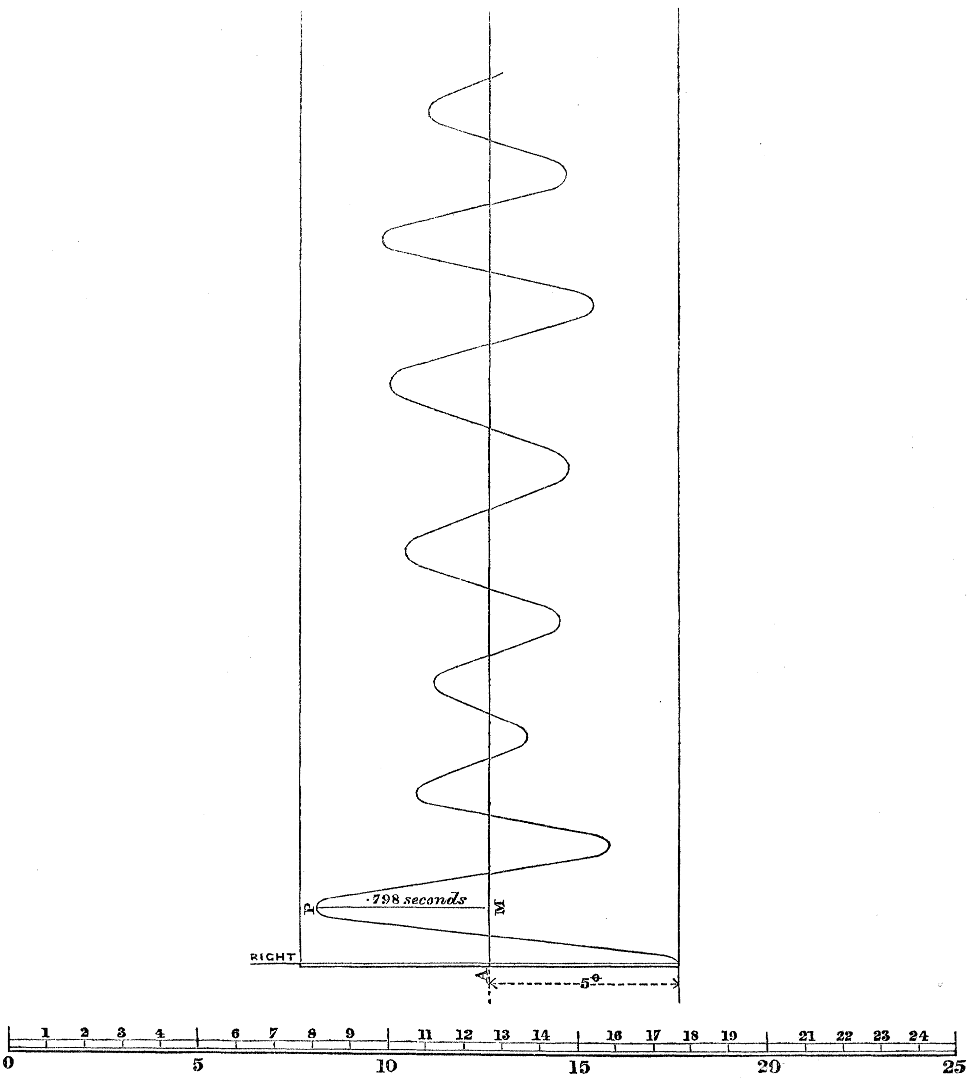


J. Basire. lith.

22. *Experiments made by Messrs.* Fincham *and* Rawson *with the cylindrical model* (Plate XLVII. fig. 2) *to verify the formula* 22, *which represents the time of the rolling of a vessel of that form.*

The following ingenious expedient for determining the time of the oscillations of a floating body was suggested by Mr. Fincham and found fully to answer the purpose.

The motion of the vessel, when floating in the tank, was limited to a rolling motion by means of upright guides (A, B, Plate XLIX.), receiving between them the extremities of a longitudinal axis fixed in the vessel so as nearly to pass through the centre of gravity. A frame, PKL, resting on two parallel rails, was made to traverse uniformly by means of clockwork, M, in the direction of this axis produced, that is, in the direction of the length of the model, and at right angles to the direction in which it rolled. This frame carried a cylindrical piece of wood, PL, having its axis in the direction of the motion of the frame, and its surface so curved that an arm, HI, fixed to the model parallel to its length and projecting beyond its extremity, should, as it



rolled, sweep parallel to the surface of the cylinder. A pencil was fixed at the extremity of this arm and pressed lightly by a spring upon the surface of the cylinder, which was covered by a piece of paper. The frame which carried this cylinder advancing in the direction of its axis, and the vessel at the same time rolling so as to sweep the arm over its surface perpendicular to that direction, a zigzag line was, by the combination of the two motions, described, as represented in the diagram on the preceding page, of which each two consecutive loops mark the beginning and end of the same oscillation, and the distance AM between them, measured in the direction in which the frame moved, shows the space traversed by it in the time occupied by that oscillation. This space being known from the experiment, and the rate at which the frame travels uniformly being also known, the time occupied in the oscillation may be determined*.

At a certain point the amplitudes of the oscillations will be observed suddenly to increase. This is due to the return of the wave created by the vessel in the act of rolling and reflected by the sides of the tank†.

Before the times of oscillation could be calculated by the formula (22.) to compare them with those determined by experiment, it was necessary that the moment of inertia W_1k^2 of the floating body should be ascertained. To ascertain it by calculation would have been a difficult task, involving in some respects an uncertain result; it was sought therefore by experiment.

With this view a knife-edge was fixed at each extremity of the cylinder, so as accurately to coincide with its axis prolonged. The vessel taken out of the water was then made to rest by means of these knife-edges on two hard steel plates, accurately adjusted to the same level, and in this position it was allowed to oscillate, the times of its oscillations being determined as before. It then became possible to determine the moment of inertia from the following well-known formula—

$$t=\pi\sqrt{\frac{k^2+h^2}{gh}}\left\{1+\left(\frac{1}{2}\right)^2\sin^2\frac{\theta}{2}+\left(\frac{1.3}{2.4}\right)^2\sin^4\frac{\theta}{2}+\ldots\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25.)$$

Two series of experiments were made, the model vessel being loaded in the one series so as to weigh 197·18 lbs. gross, and in the other so as to weigh 255·43 lbs. Having obtained from the above formula a mean value of k in respect to each of

* The apparatus, and the method of experimenting with it, are more fully described in the Appendix. It will be observed that in the diagram the amplitudes of the successive oscillations are shown to have diminished, up to a certain point. This diminution was found in a great number of experiments to take place with remarkable uniformity. The uniform diminution of the amplitude of oscillation as the body comes to rest, is of course due to the absorption of the *vis viva* of the rolling body by the water which it puts in motion, to the friction of the water which adheres to it on the rest of the water, to the resistance of the air, and to the friction and abrasion of the parts of the rolling body itself in the act of rolling. Mr. RAWSON has made some experiments on this subject, the results of which he communicated to the British Association for the Advancement of Science.

† The precision with which the instant of the return of this wave and its progress are indicated, seems to show this method of experimenting to be well-suited to determining the velocities of waves.

these series of experiments, Mr. Rawson verified these mean values by calculating from them the times of oscillation by the following equivalent formula:—

$$t=\sqrt{\frac{k^2+h^2}{gh}}.\mathrm{F}c\frac{\pi}{2},$$

which I had previously communicated to him, in which $\mathrm{F}c\frac{\pi}{2}$ represents that complete elliptic function of the first order whose modulus is c. The results of these experiments and calculations are given in the following Table:—

Gross weight of model.	Amplitude of oscillation on either side of vertical.	Time of oscillation by experiment.	Value of $\sqrt{k^2+h^2}$ by formula $\frac{\sqrt{ag}.t}{\pi(1+f\theta)}$ (see Appendix).	Mean value of k deduced from the preceding column.	Time of oscillation calculated from the formula $t=2\sqrt{\frac{k^2+h^2}{gh}}.\mathrm{F}\left(c\frac{\pi}{2}\right)$ with the preceding value of k, verifying that value.
lbs.	° ′	″	in.	in.	″
	5 0	·8190	11·983	......	·8348
197·18	7 30	·8905	12·562	10·465	·8352
	10 0	·7930	11·554	......	·8359
	2 30	·7280	11·070	......	·7280
255·43	5 0	·6500	10·451	9·3294	·6500
	7 30	·7800	11·784		
	10 0	·6760	11·706		

Table of the times of the oscillation in water of a model vessel as determined,—1st, by the experiments of Messrs. Fincham and Rawson; and 2ndly, by the formula (22.),

$$t=\frac{\pi k}{\sqrt{gh_1\left(1\mp\frac{\mathrm{W}_2h_2}{\mathrm{W}_1h_1}\right)}}\left\{1+\frac{4h_1^2+k^2}{4k^2}\sin^2\frac{1}{2}\theta_1\right\},$$

Constants of formula.	Angle of oscillation θ_1.	Time of oscillation in seconds by formula.	Time of oscillation in seconds by experiment.
Weight of model $=197·18$ lbs.	5°	·75637	·798
$\mathrm{W}_2=0$			
$k=10·465$ in.	10	·75840	·798
$h_1=4·9$ in.	15	·76172	·840*
$t=·75668\left\{1+·46924\sin^2\frac{1}{2}\theta_1\right\}$	20	·76641	·784
Weight of model $=255·43$ lbs.	5	·61094	·658
$\mathrm{W}_2=0$			
$k=9·3294$ in.	10	·61289	·665
$h_1=5·97$ in.	15	·61471	·616
$t=·61033\left\{1+·56021\sin^2\frac{1}{2}\theta_1\right\}$	20	·64811	·602

23. *General considerations applicable to the stability of floating bodies.*

An exceedingly small pressure is sufficient to move a floating body from its position of stable equilibrium. As the inclination becomes greater the pressure must be increased until it attains, in a given position, its maximum value. Now if instead of this variable and continually increasing pressure we conceive a constant one to

* This experiment was faulty.

have been applied, less than the maximum one spoken of above, but considerably greater than that necessary to produce the initial motion, it is not difficult to see that this lesser pressure, by reason of its continued action on the body, may be sufficient to carry it *through* the position in which it would have required the *maximum* pressure to have *held* it; for in all the positions in which this pressure exceeded that necessary to move the body, the excess of its work over that of the resistance is accumulated under the form of *vis viva*, and in all those in which it fell short of that pressure, the work thus accumulated comes to its aid, and carries the body forward*.

24. To move a body from a given position into any other, it is not necessary that the work of the forces whence this change of position results should continue to be done upon it during the whole period of such change. They may be in the nature of pressures whose action ceases when they have communicated the *vis viva* necessary to continue the motion until the second position is attained, as in the case of the impact of one body on another, or of a gust of wind acting on a floating body, or of a blow of the sea.

In all these cases the excess of the work of the disturbing forces over that of the

* As an illustration of this principle, let us take the analogous case of a cylindrical body (fig. 7)—whose radius CD is represented by a—rolling on a horizontal plane, and to simplify the investigation let us suppose the force P, which causes it to roll, to be applied horizontally to the axis c. Let the weight of the cylinder be represented by W, and suppose it to be so loaded that the distance of its centre of gravity from its axis may be $CG(=c)$. Let the cylinder be made to roll through an angle (θ) from the position in which it would rest, and in which CG is vertical. The work done upon it by P, whilst it thus rolls, is represented by $Pa\theta$, and the work opposed to its rolling by the weight of the cylinder, is $Wc \text{ vers } \theta$,

$$Pa\theta - Wc \text{ vers } \theta$$

is therefore the excess of the work of the forces whose tendency is in the direction of the motion over the work of those whose tendency is in the opposite direction. This excess represents one-half the *vis viva*, and in the extreme position into which the cylinder rolls and from which it begins to roll back, this vanishes, so that in this position

$$Pa\theta - Wc \text{ vers } \theta = 0.$$

If this position be the inverted position of the body $\theta = \pi$, and

$$P = \frac{2Wc}{\pi a}.$$

Now let us suppose that P, instead of being a constant pressure, had been made so to vary that in every position into which the body rolled it was only just sufficient to make it roll *slowly* out of that position. The maximum pressure P′, applied under these circumstances, would have been that corresponding to the position in which CG is horizontal, and would have been represented by the formula

$$P' = \frac{Wc}{a},$$

whence it follows that

$$P = \frac{2}{\pi} P' = \cdot 6366 \, P';$$

so that the pressure, which, being continually applied the same in amount, would be sufficient to overturn the body, is less than two-thirds of that which must be exerted in its position of maximum effort, the lesser but constant pressure accumulating in its previous position, an excess of work which is sufficient to carry the body through and far beyond that position in which it would have required the greater pressure to have held it.

resistances accumulates, so as to be represented at any time by half the *vis viva* of the body, and it is the work accumulated under this form and in this amount, which, when the operation of the disturbing forces is withdrawn, carries the body forwards.

25. If, in the case of an oscillatory motion, the force which causes that motion be intermittent, and if the periods of this intermission so coincide with those of the body's oscillation that the force is withdrawn when any oscillation in its direction is completed, and renewed when the next oscillation in *that direction* begins, it is evident that the *vis viva* created by it in all such successive oscillations will be accumulated, and the amplitudes of the oscillations rendered, in succession, greater than one another, so that by the intermittent action of a small force which so synchronizes with the oscillations of the body on which it is made to act, a great inclination of its position from that of its equilibrium may eventually result*. This might for instance be produced, in respect to a floating body, by the action of gusts of wind or blows of the waves, repeated at stated intervals.

If the periods at which the intermittent work is done upon the body, instead of synchronizing with the commencement of each oscillation and thus favouring the motion, had so occurred as to oppose it at stated intervals (which may or may not synchronize with the body's own oscillations), it is evident that a complicated motion dependent on these several conditions will result, in which the oscillations will be repeated in cycles.

* As an illustration of this, let us suppose that in case of the cylinder (fig. 7), the motive force. instead of being applied horizontally to the axis, is a weight w applied to the point D on its surface when the cylinder was inclined at the angle θ to its position of stable equilibrium; and that when the point D had, by the rolling of the cylinder, been made to descend until it again touched the plane, the weight w was withdrawn, the cylinder completing its oscillation on the other side of the vertical by reason of the *vis viva* thus communicated to it.

On its return it will oscillate (the resistance being neglected) through the same angle, on the side of the vertical from which it started, as it completed on the opposite side. Let the weight w be then placed again on the point D, and taken off a second time when this point comes in contact with the plane; and so continually, a rocking motion of increasing amplitude being produced by the alternate placing and withdrawal of the weight until at last the position of CG is reversed and the cylinder is overturned.

Let θ_1, θ_2, $\theta_3 \ldots \theta_n$ be the angles on either side of the vertical through which, in its successive oscillations, the cylinder is made to roll, then shall we obviously have the following relations:—

$$(\mathrm{W}c+wa)\operatorname{vers}\theta=\mathrm{W}c\operatorname{vers}\theta_1,$$
$$(\mathrm{W}c+wa)\operatorname{vers}\theta_1=\mathrm{W}c\operatorname{vers}\theta_2,$$
$$\&\text{c.} \quad = \quad \&\text{c.}$$
$$(\mathrm{W}c+wa)\operatorname{vers}\theta_{n-1}=\mathrm{W}c\operatorname{vers}\theta_n.$$

Multiplying these equations together,

$$(\mathrm{W}c+wa)^n\operatorname{vers}\theta=(\mathrm{W}c)^n\operatorname{vers}\theta_n.$$

But $$\theta_n=\pi, \quad \therefore \operatorname{vers}\theta_n=2 \text{ and } \operatorname{vers}\theta=2\sin^2\frac{1}{2}\theta,$$

$$\therefore (\mathrm{W}c+wa)^n\sin^2\frac{1}{2}\theta=(\mathrm{W}c)^n,$$

and $$w=\frac{\mathrm{W}c}{a}\left\{\left(\operatorname{cosec}\frac{1}{2}\theta\right)^{\frac{2}{n}}-1\right\}.$$

26. If, when the work of any disturbing force begins to be done upon a floating body, it be already acted upon by some other force, which has caused it to incline from the position in which it would otherwise rest through some given angle α, as in the case of a squall or a heavy sea striking a ship already inclined by the action of a steady wind upon her beam; representing by θ the additional inclination given to it by the action of the disturbing force, whose work, in giving it this inclination, is represented by $u(\theta)$, and representing the work done through this same angle θ by the forces originally impressed, and still acting, upon it by $U(\theta)$, we have

$$Wc\{\text{vers}\,(\alpha+\theta)-\text{vers}\,\alpha\}=U(\theta)+u(\theta).$$

Differentiating and transposing,

$$\delta\theta=\{\delta U(\theta)+\delta u(\theta)\}\frac{1}{Wc\sin(\alpha+\theta)}.$$

The small additional angle $\delta\theta$ through which the body is made to roll by the application to it of a given small additional amount of work,

$$\delta U(\theta)+\delta u(\theta),$$

varies therefore inversely as the sine of the inclination, and is greatest in the position nearest to the vertical position; or, in other words, the body sustains in an inclined position a less change of that position by the application of a given disturbing force than it does in a vertical position. It yields most readily to the action of any disturbing force in its vertical position, and the further it is made to deviate from this position (within certain limits and under certain conditions), the more resolutely does it oppose any further deviation. This explains the liability of a ship to rolling when sailing before the wind, and her stiffness in the water when close hauled.

Conclusions applicable to Ship-building.

27. To make an alteration in the angle through which a ship rolls, it is necessary to elevate or to depress her weights. In the former case she will roll through a greater, and in the latter through a less angle. It does not alter the amplitude of rolling to move the weights horizontally, but only the time of rolling, provided the trim of the ship remain unaltered; for this does not alter the position of the centre of gravity of the ship or of the displaced fluid, and it is upon these that the stability of the ship depends (Art. 7 and equation 18.).

28. When a ship's motion is only a rolling motion, or about an axis parallel to her length, the position of that axis is, at any instant, determined by the intersection of a horizontal line through her centre of gravity, and a vertical line through the centre of gravity of her plane of flotation at that instant (Art. 19).

29. A ship should be so constructed that the centre of gravity of that plane of flotation, whose boundary is the load water-line, may be vertically above the centre of gravity of the ship. If this be not the case, the pressure of the additional water displaced by any vertical oscillation of the ship, acting obviously at the centre of gravity

of that plane and not in the same vertical with the centre of gravity of the ship, will cause, in addition to that vertical oscillation, a pitching or a rolling motion.

30. All the planes of flotation of the ship, when it is made to roll through small angles from its vertical position, should have their centres of gravity in the midship section (supposed to be that which contains the centre of gravity of the ship), and the centre of gravity of the displaced fluid should also always remain in this plane; for if that be not the case, it is obvious, from Art. 28, that the axis about which the vessel rolls cannot be parallel to its length, so that every rolling must be accompanied also by a pitching motion.

31. The angle through which a ship rolls under the action of a gust of wind, is essentially different from that at which it would be held inclined by the steady action of the same force of the wind, so that when the inclination which a given pressure would give to the ship, if applied to the centre of effort of the wind on the sails, is calculated (as is customary) by the formula known as that of ATWOOD, it is erroneous to conclude that the vessel, if subjected suddenly to that force of the wind, would incline only through that angle.

The experiments of Messrs. FINCHAM and RAWSON, of which the results are stated in the Table, p. 615, show that it will roll through nearly double that angle, confirming, in this respect, the deductions of theory (Art. 2).

Neither can calculations, made by means of the theorem of ATWOOD, as to the respective pressures which would hold different ships inclined at the same given angle, be considered as determining with certainty their *relative* stabilities in respect to rolling; for the amplitude of each oscillation depends upon the stiffness of the vessel, not only in respect to that given inclination, but upon its stiffness at every other inclination which it must have passed through to reach that angle, and at every inclination which by reason of its acquired momentum it may pass through afterwards. The same observation is applicable to the effect of disturbances in the water-line and to blows of the waves.

32. All the causes which produce the rolling motion of a ship, whether they be gusts of wind, disturbances of the water-line, or blows of the waves, are measured in their effect upon it under the form of *work* (travail), so that according as a ship requires a greater or a less amount of work to be done upon her to cause her to roll through a given angle, there is a greater or less probability that, when at sea, she will roll through so great an angle as that. The angle through which the ship will be made to roll under the action of a given amount of *work*, is measured (Art. 7) by the product of her weight, by the difference or the sum of the vertical displacements of her centre of gravity, and the centre of gravity of her immersed part whilst she is in the act of rolling through that angle; the difference being taken or the sum according as these centres of gravity both ascend, or as the one ascends and the other descends; or in other words, it is *the work necessary to raise the vessel bodily through the difference or the sum of these vertical displacements.*

33. If therefore some existing vessel were fixed upon whose qualities in respect to rolling were well known, and if it were determined by calculation from this theorem what amount of work must be done upon that vessel to make it roll through some *given angle*; and this amount of work being so determined in respect to that existing ship, if, before all other ships of the same class were built, it were determined by a similar calculation, made from the drawings of those ships, whether a greater or a less amount of work would be necessary to make *them* roll through the same angle, then it would be known whether *these* ships would, under the like circumstances, roll more or less than *that* ship, and the forms proposed to be given them might be adopted or might be altered accordingly.

I conceive that by this means, if duly applied, great certainty might be given to the construction of ships in respect to rolling, and of course to pitching, for the same principles which apply to the one apply also to the other, with no other difference than in the direction in which the inclination is supposed to take place.

34. The force of the winds and waves, to the action of which a vessel is liable, may be supposed to vary as the surface she opposes to them, that is, to the area of her sails and the superficial dimensions of her hull. In vessels geometrically similar these vary as the squares of any of their similar linear dimensions, their lengths for instance. On the other hand, the weights of such vessels, supposed to be similarly loaded, varying as the cubes of their lengths; and the depths of their centres of gravity, and of the centres of gravity of their immersed parts, varying as their lengths; their dynamical stabilities, with reference to a given inclination, vary as the fourth powers of their lengths. Since, then, in reference to vessels thus geometrically similar, the disturbing forces, to the action of which they are subject, vary as the squares of their lengths and their stabilities as the fourth powers, it follows that their actual steadiness in the water will vary as the *squares* of their lengths, the greater vessel being more steady than the less in this proportion.

35. The expedients which I have pointed out for so designing a ship as to satisfy the conditions of easy rolling and pitching, suppose a knowledge of the exact position of the centre of gravity of the ship and of the centre of gravity of her displacement. The determination of these however is no new question; a knowledge of them has always been considered necessary to the skilful building of a ship, and the methods given for that purpose in books on ship-building are sufficiently accurate for the purpose, if the data are to be relied upon; and if not, nothing is required but the labour to determine these data.

36. That form of vessel in which the surfaces subject to immersion and emersion, when intersected by planes perpendicular to the vessel's length, *have circular* sections, having their centres in a common axis, is, *cæteris paribus*, eminently a *stable form*; because in a vessel of such a form (Art. 12.) the centre of gravity of the portion of the displaced fluid which is included within the solid of revolution (ATB figs. 3, 4) formed by all these circular sections, does not in the act of rolling *rise*.

If it be not practicable to give to the vessel, throughout its whole length, a form subject to these conditions, this is practicable with regard to the midship section, which is the governing section.

37. Of vessels having this general form, those are, *cæteris paribus*, the most stable in which the circular area, when completed, includes entirely the corresponding section of the ship (as shown in fig. 4), because, in respect to these ships, the centre of gravity of the displaced fluid *descends* as the ship rolls (Art. 13.); and when this is the case, the work necessary to incline the ship through any given angle (which measures its dynamical stability) is equal to that necessary to raise it bodily through a height equal to the *sum* of the vertical displacements which its centre of gravity, and the centre of gravity of the displaced fluid, suffer in that inclination, whilst in the opposite case, represented in fig. 3, it is equal to the work necessary to raise it through the *difference* of those heights*.

Of the class of vessels represented in fig. 4, the stability of those is the greatest, other things being the same, in respect to which the space STD between the circumference of the circular area and the hull of the vessel is the greatest.

38. It is not necessary to these results, taken in a general sense, that the sections of the vessels should be accurately circular as to their parts subject to immersion and emersion. If on the midship section of a ship a circle can be described, having its centre in the vertical axis of the section so as nearly to coincide with the parts of the periphery subject to immersion and emersion, then the ship may be distinguished as to whether it belongs to the class in which the centre of gravity of the immersion ascends or descends, by observing whether this circular area is wholly or only partly included within the section. The midship sections of Her Majesty's ships Vanguard, Bellerophon and Canopus present illustrations of this principle. They are represented in figs. 8, 9, 10. It will be observed, that if a circular area be struck on each section according to the conditions stated above, that area will in the Bellerophon and Canopus entirely include the corresponding sections of the hulls of the two ships, a wider space intervening between the two areas in the Canopus than the Bellerophon. Other things being the same, the former ought therefore to be the more stable ship. In the Vanguard the circular section does not wholly include that of the hull. This should therefore be the least stable ship of the three. These conclusions are, I believe, in accordance with the known qualities of the ships. If there be any ship whose midship section resembles that represented in fig. 3, the centre of gravity of its displacement will ascend in the act of rolling, and, *cæteris paribus*, it cannot but be an unstable ship.

* The hull may be so shaped as to cause the centre of gravity of the vessel to descend in the act of rolling. In this case ΔH_1, equation 6, must be taken negatively. If the centre of gravity of the immersed part ascends, $U(\theta)$ will in this case be negative, and the position will be one of unstable equilibrium. If the centre of gravity of the immersion descends, ΔH_2, in equation 6, must be taken positively; and if it exceeds ΔH_1, the equilibrium will be stable.

In all these comparisons it has been supposed that the ships do not otherwise differ as it regards their stability, than in their approximation to the typical forms represented in figs. 3 and 4. It has been supposed therefore (see equation 17) that the depths of their centres of gravity and of the centres of gravity of the fluid they displace in a vertical position are the same, and that the moments of inertia of their planes of flotation are equal. This cannot be the case; and it is impossible to know to what extent this error in the hypothesis may in any particular case affect the measure of the dynamical stability, except by calculating it by formula 17. The problem is far too complicated to render the application of any general principle—except with this precaution—safe. It is not—in this respect as in others—more practicable to dispense with the resources of mathematical reasoning and of calculation, in building the ship, than, after she is built, in sailing her.

39. It is a deduction of theory (equation 24.), and is confirmed by experiment, that, within the ordinary limits of rolling, the *Time* of a vessel's rolling is independent of the angle through which it rolls, being dependent only upon the form of the vessel, its weight, the position of its centre of gravity, and its moment of inertia about an axis passing through that point (see equation 24); so that every different vessel, when loaded in a given manner, has *a time of rolling proper and peculiar to it*, and which may be said to *characterize* it. And the same is true of the time of pitching.

40. This time of the oscillations of the ship may have such a relation to the times of oscillation of the waves* as to cause the blows of the sea to be received by the ship at those instants when they will produce the greatest effect on the amplitude of her oscillations (Art. 25.), and thus a little sea may, under certain circumstances, produce very heavy rolling.

If there be not this relation between successive oscillations of the ship and of the waves, then there will be, during a certain number of oscillations, an antagonism of the two, until the times of the one class of oscillations have so gained upon those of the other as to bring about an interference. The vessel will then probably be comparatively at rest. Then will follow a series of oscillations of the waves and the ship, which will in various degrees concur to produce heavy rolling, until it reaches a maximum, when the same cycle of changes will be gone through again†.

41. The straining of the ship in the act of rolling, is dependent upon the time of its oscillations. This straining takes place in every part of it, but more particularly (by reason of their elasticity) in the masts. When the rolling begins, the higher parts of the masts, by reason of their inertia, remain behind the lower portions, and the masts bend; as the rolling proceeds they receive an independent motion

* If the ship be under sail, the rate at which she sails and the distances of the waves from one another, measured in the direction in which she sails, are also among the conditions on which her rolling depends.

† This cyclical antagonism and concurrence of the independent oscillations of the waves and the ship will interfere, to a certain extent, with the tautochronism of the ship's oscillations.

from their elasticity, influenced, besides this cause, by their weight and length, and this motion assumes the form of an independent oscillation, affecting more or less the oscillatory motion of the vessel itself.

If, at the instant when the ship would otherwise begin to roll back, the elasticity of its masts is in the act of carrying them forwards, there will be a violent strain of the ship, which would not take place in a ship so constructed or so loaded as to create that synchronism of the independent oscillations of the vessel and its masts, which is implied in the fact that she does not strain herself in rolling.

42. The properties of different ships with regard to the strain they suffer in rolling, and whether, in respect to the seas which are of most frequent occurrence, they roll easily or heavily, are probably well known, so that it would be possible to fix upon some ship of each class which might in this respect serve as a standard with which other ships of that class might be compared. The time of oscillation proper and peculiar to this ship might also be ascertained by experiment. It would then become possible to build and load all other ships of that class so as to oscillate in the same time as that ship, and thereby ensure, supposing them to be masted in the same way, very nearly the same qualities in regard to the strain they suffer in rolling.

The form of the ship so constructed need not however in any respect resemble that of the standard ship; all that is required is, that—in respect to the dimensions of its parts and the distribution of its weights when loaded—it should satisfy the conditions implied in assigning a given value to $t(\theta_1)$ in equation 24. In all other respects full latitude is allowed to the builder*.

43. I have shown (Art. 19.) that the axis about which the ship may at any instant be conceived to roll is perpendicular to two straight lines, one of which is drawn horizontally from the vessel's centre of gravity parallel to the direction in which it is rolling, and the other vertically through the centre of gravity of its plane of flotation at that instant.

Its *vis viva*, when rolling or pitching, is therefore greater as its weights are placed at a greater distance from this axis, and less as they are nearer to it.

Whence it follows as a general principle (equation 24.), that the ship is made to roll more slowly by moving its weights to a greater distance from this axis, as when the yards are run up; and more quickly by bringing them nearer to it, as when the guns are run back in a heavy sea to diminish the strain on the ship's timbers.

44. The form under which H_2 enters equation 24, shows that the greater the depth of the centre of gravity of the ship's displacement the more slowly (other things being the same) will she roll, provided that she be a ship of that class of which fig. 3 is the type, in which this quantity (*i. e.* the depth of the centre of gravity of the displacement) diminishes as the vessel rolls; but that if, on the contrary, she belongs to

* Taking a ship, whose form may indeed (within certain assignable limits) be any whatever, it is in his power, guided by that equation, so to *load* it as that it shall oscillate in the same time with any other ship whose form may be in all respects different.

the class of which fig. 4 is the type, the centre of gravity of whose displacement descends in the act of rolling, she will roll the faster as this centre of gravity is seated deeper in her hull.

45. It may be received as a general rule that (other things being the same) vessels of the class fig. 4 roll more quickly than those of the class fig. 3, but do not roll so far. So that unless some special provision be made for that end, stability in rolling may not, and will not probably be obtained except at the expense of quick rolling.

46. The form under which A enters equation 24, shows that breadth of beam is not *of itself* conducive to slow rolling, and that it can only tend to it indirectly, by carrying the ship's weights further from the axis about which it rolls, and thereby augmenting the quantity k, which represents in the formula the moment of inertia of the ship, and is the ruling element of the time. The form under which W_1 enters the equation shows heavy loading to be conducive to slow rolling.

47. It would be unsafe however to be guided in the construction of a ship by any one of these considerations taken separately from the rest, for there is scarcely any element of the discussion which can be changed without bringing about a change in an opposite direction in some other.

What will be the total result of any proposed change, and by what means the whole object sought by it is to be accomplished, can only be determined by that complete mathematical discussion of the question in all its elements, the principles of which it has been the object of this paper to develope, and which the formulæ contained in it afford a means of applying.

December 1, 1849.

APPENDIX.

Experiments on the Dynamical Stability and the Oscillations of Floating Bodies. By JOHN FINCHAM, *Esq., Master Shipwright in Her Majesty's Dockyard, Portsmouth, and* ROBERT RAWSON, *Esq.*

Experiments necessary to verify the formula 6,

$$U(\theta)=W(\Delta H_1-\Delta H_2),$$

which represents the work done in deflecting a floating body through an angle such that the height through which the centre of gravity of the floating body is raised shall be ΔH_1; and the height through which the centre of gravity of displacement is raised, shall be ΔH_2.

For this purpose two models were made, such that their sections were uniform throughout the whole length of the model: the section of one was a triangle, and of the other a circle (see figs. 1, 2).

If the reader will refer to Art. 10 for a general description of the apparatus, the following explanation of the drawing fig. 1 will be sufficient to show how the experiments have been conducted. It represents a section of the model; TK is the yard by means of which the deflecting weight hanging at Q, deflects the model from the upright position. AC is water-line in the upright position, and BD that in the extreme position into which it rolls; EF that in the position in which it finally rests.

In the first place, the model is adjusted by means of moveable weights, until the water-line AC is parallel to the upper side, LN, of the model; and then a string, SK, is fixed at S and K, so that when the deflecting weight is placed at Q no effect is produced by the deflecting weight on the model until the string SK is cut (RS is a fixed beam independent of the model).

When the string SK is cut there is an extreme deflection where the water-line becomes BD, and a permanent deflection where the water-line becomes EF.

These lines are determined in the following manner: PX is a thin graduated scale, fixed at P at right angles to the arm of the lever TK, and having a strip of prepared paper fixed upon its surface, which shows distinctly, by the depth to which it is wetted when the vessel rolls, a point in the extreme position of the water-line. Two other points in this position of the water-line are determined by means of scales similarly applied to L and N.

When any two of these three points are observed, a section of the model being drawn of the half-size on a drawing-board, we could set off upon it the distances P*p*, L*l*, N*n*, and thus draw in the water-line BD. There is no occasion to use prepared paper to show the water-lines, excepting in the ultimate deflection.

The paper which we used, and which answered the purpose admirably, was nothing more than common writing-paper rubbed over with a little colouring, in order to take off from the surface of the paper any oily matter which might prevent the water's making a distinct mark upon it.

This means was adopted as the best means, after several other expedients had been tried with partial success.

The centre of gravity G was determined by observing the permanent water-line EF in a number of deflections by means of various deflecting weights. The position of this water-line, for any given deflecting weight, being set out on the drawing, we were enabled, knowing the weight of the vessel, to determine the position of the centre of gravity, by well-known principles of statics, with no other aid than that of the scale and compasses. Suffice it to say, that great care was taken to get this point.

The points H and *h*, which are the centres of gravity of the part immersed in the vertical position of the vessel and in the position into which it rolls, were determined from the known property of the centre of gravity of a triangle in fig. 1, and from the common formula for determining the centre of gravity of the segment of circle in fig. 2.

The drawings were made to half size from the following Table, which was filled up during the time the experiments were in operation: this circumstance enabled us obtain all the data required for our computations with extreme exactness.

Cylindrical Model.

Model at rest, without the deflecting weight.			Model at rest, with the deflecting weight.			Model at ultimate deflection, with deflecting weight.			Weight of the model and deflecting weight.		Distance from centre of model to P and Q.		
L.	L*k*.	N*s*.	P*o*.	L*q*.	N*r*.	P*p*.	L*l*.	N*n*.	Deflecting weight.	Weight of model.	P.	Q.	
									lbs. Av.	lbs. Av.	ft. in.	ft. in.	
	$8\frac{1}{16}$	$8\frac{1}{16}$	...	6	$10\frac{1}{8}$	$31\frac{5}{16}$	$4\frac{3}{16}$	...	1·957	197·18	3 $6\frac{1}{4}$	5 $4\frac{5}{8}$	(2.)
	8	$8\frac{1}{8}$	...	6	$10\frac{1}{8}$	$31\frac{5}{16}$	$4\frac{1}{4}$	...	......	......	3 $6\frac{1}{4}$	5 $4\frac{5}{8}$	
				$6\frac{1}{2}$	$9\frac{1}{2}$	......	...	...	......	......	3 $6\frac{1}{4}$	4 $0\frac{5}{8}$	
	6	$6\frac{1}{4}$	...	$4\frac{3}{4}$	$7\frac{3}{16}$	$34\frac{3}{16}$	$3\frac{3}{4}$	...	1·957	255·43	3 $6\frac{1}{4}$	5 $4\frac{5}{8}$	(3.)
				$5\frac{1}{6}$	7	......	...	...	......	......	3 $6\frac{1}{4}$	4 $0\frac{5}{8}$	
	6	$6\frac{1}{16}$	31	$2\frac{1}{2}$	$9\frac{7}{16}$	$20\frac{7}{8}$	$0\frac{5}{8}$	...	5·0285	255·43	3 $6\frac{1}{4}$	5 $4\frac{5}{8}$	(4.)
			$33\frac{1}{2}$	$3\frac{7}{16}$	$8\frac{9}{16}$	21	...	...	......	......	3 $6\frac{1}{4}$	4 $0\frac{5}{8}$	
	$8\frac{1}{8}$	$8\frac{1}{8}$	34	5	$11\frac{3}{16}$	$24\frac{5}{16}$	$2\frac{3}{16}$	...	2·8225	197·18	3 $6\frac{1}{4}$	5 $4\frac{5}{8}$	(5.)
			$36\frac{1}{4}$	$5\frac{3}{4}$	$10\frac{5}{16}$	...	...	...	......	......	3 $6\frac{1}{4}$	4 $0\frac{5}{8}$	

The position of the water-line corresponding to the extreme position into which the model rolled being thus determined in every experiment, the corresponding displacement was also known. In fig. 1, this displacement having a triangular section *eMf*, its centre of gravity *h* could readily be determined by construction. In fig. 2, the displacement having a circular section, its centre of gravity *h* could be determined by known rules. Drawing a perpendicular *hc* from *h* upon BD, this line measures the depth of the centre of gravity of the immersion in the extreme position into which the vessel rolls, and H*a* measures it in the vertical position; therefore the difference of these lines measures its elevation in the act of rolling. In like manner the perpendicular G*b* measures the depth of the centre of gravity of the *model* when the water-line was BD, and G*a* was its depth in the vertical position, therefore the difference of these lines is its elevation in the act of rolling.

Thus the elevations of the centre of gravity of the model and of the centre of gravity of the displaced fluid, in the act of rolling, are found, and its weight being known, the work which must have been done upon it to cause it thus to roll may be determined according to equation 6.

But the work actually done upon it by the deflecting weight may, in like manner, be determined; for if a perpendicular Q*o* be drawn from Q on BD, the length of this line will be the height of Q when the water-line was BD, and Q*m* was its height in the vertical position, therefore the difference of these lines is the space through which the deflecting weight has descended vertically; and the product of this distance of the deflecting weight gives the work actually done by it upon the rolling body. This amount of work ought, by the theory, to be the same with that found as above from equation 6; and the comparison of results thus obtained, by theory and experiment, constitutes a verification of the formula, and is given in the Table, p. 615.

A single example will show the way in which the calculations were made.

In the triangular model, Experiment 1, the following dimensions were measured in inches:—

$$Ga=1{\cdot}75,\quad Gb=1{\cdot}05,\quad Ha=2{\cdot}75$$
$$hc=2{\cdot}24,\quad Qm=15{\cdot}2,\quad Qo=3{\cdot}9$$

∴ elevation of centre of gravity of model, in feet $=\frac{1{\cdot}75-1{\cdot}05}{12}=\Delta H_1$

elevation of centre of gravity of displaced fluid, in feet$=\frac{2{\cdot}75-2{\cdot}24}{12}=\Delta H_2$

weight of model in lbs. . . . . $=33{\cdot}8626 = W$.

∴ by equation 6, $U(\theta)=\left\{\frac{(1{\cdot}75-1{\cdot}05)-(2{\cdot}75-2{\cdot}24)}{12}\right\}\times 33{\cdot}8626=\cdot 5361$.

Also vertical descent of deflecting weight, in feet$=\frac{15{\cdot}2-3{\cdot}9}{12}$,

deflecting weight Q in lbs.$=\cdot 5485$;

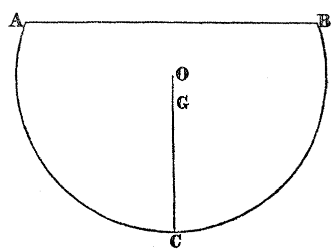
∴ by experiment $U(\theta)=\left\{\frac{15{\cdot}2-3{\cdot}9}{12}\right\}\times\cdot 5485=\cdot 5165$.

The computations on both these models, made in accordance with formula 6, agree with the experiments.

All the experiments show what we conceive to be important, that the same moment of force which is necessary to maintain a vessel in a position θ at rest, will deflect the vessel through nearly twice θ when made to act upon a vessel which is not deflected. ATWOOD's statistical stability therefore appears to us to be of little use, so far as the rolling of the vessel is concerned.

Experiments on the Time of Oscillation.

To find the moment of inertia of a cylindrical model, whose radius is 13·5 inches, weight 197·18 lbs. avoirdupois, round an axis passing through the axis of the cylinder.



ABC is a vertical section of the cylinder passing through its centre of gravity G.

It is required to find the moment of inertia of the cylindrical model, about an axis passing through O, the centre of the cylinder at right angles to the section ABC.

For this purpose the model was made to oscillate out of the water, upon knife-edges passing through O and parallel to the sides of the cylinder. Several experiments were carefully made, and the times of oscillations were observed through the amplitude of 5°, $7\frac{1}{2}$° and 10°. The apparatus by means of which the times were observed, enables us to measure the times of oscillation to an exactness of ·013 of a second, or in round numbers to the one-hundredth part of a second of time.

If T= time of oscillation of a simple pendulum,
2θ= angle of amplitude of oscillation,
r= length of pendulum,
g= 32·19084 feet,
π= 3·1415927, &c. &c.,
we have

$$T=\pi\sqrt{\frac{r}{g}}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{\text{vers}\,\theta}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{\text{vers}\,\theta}{2}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{\text{vers}\,\theta}{2}\right)^3+\text{\&c.*}\right\}.$$

Put R= radius of oscillation of a compound pendulum.

Then the time of oscillation of the compound pendulum will be the same as in the case of the simple pendulum, if we put R instead of r in the above formula.

$$T=\pi\sqrt{\frac{R}{g}}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{\text{versin}\,\theta}{2}\right)+\left(\frac{1.3}{2.4}\right)^2\left(\frac{\text{versin}\,\theta}{2}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2\left(\frac{\text{versin}\,\theta}{2}\right)^3\text{\&c.}\right\}. \quad . \quad . \quad (26.)$$

We may here observe that

$$R=\frac{k^2+h^2}{h},$$

where k= the radius of gyration about the centre of gravity, h= the distance from the centre of suspension to the centre of gravity of the model.

Since

$$\frac{\text{versin}\,\theta}{2}=\frac{1-\cos\theta}{2}=\sin\frac{\theta}{2},$$

we shall have

$$T=\pi\sqrt{\frac{k^2+h^2}{}}\left\{1+\left(\frac{1}{2}\right)^2\sin\frac{\theta}{2}+\left(\frac{1.3}{2.4}\right)^2\sin^4\frac{\theta}{2}+\left(\frac{1.3.5}{2.4.6}\right)^2\sin^6\frac{\theta}{2}+\text{\&c.}\right\}.. \quad . \quad (27.)$$

If we put

$$f(\theta)=\left(\frac{1}{2}\right)^2\sin^2\frac{\theta}{2}\left\{1+\left(\frac{3}{4}\right)^2\sin^2\frac{\theta}{2}+\left(\frac{3.5}{4.6}\right)^2\sin^4\frac{\theta}{2}+\text{\&c. \&c.}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (28.)$$

equation (4.) will become

$$T=\pi\sqrt{\frac{k^2+h^2}{gh}}\,\{1+f(\theta)\}.$$

From which equation we obtain

$$\sqrt{k^2+h^2}=\frac{\sqrt{gs}}{\pi\{1+f(\theta)\}}\times T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29.)$$

If T_1, T_2, T_3 be the time of the first, second and third oscillations during the time that θ and s remain constant, we shall have

$$n\sqrt{k^2+h^2}=\frac{\sqrt{gs}}{\pi\{1+f(\theta)\}}\left\{T_1+T_2+T_3+\text{\&c. to } n \text{ terms}\right\},$$

$$\therefore\ \sqrt{k^2+h_2}=\frac{\sqrt{gs}}{n\pi\{1+f(\theta)\}}\left\{T_1+T_2+T_3\,\text{\&c. to } n \text{ terms}\right\}.. \quad . \quad . \quad . \quad . \quad . \quad (30.)$$

* See Poisson's Traité de Mécanique, p. 348.

For very small angles of amplitude $f(\theta)$ is very small, and may be entirely neglected, showing that for small angles the oscillations are independent of the angles.

The value of $\sqrt{k^2+h^2}$ was in the first place calculated in respect to *one* of the quantities T_1, T_2, &c., then in respect to *two*, and so on, to *four*. The mean of these four values being then taken in respect to each experiment, a final mean was taken in regard to all the experiments. The data and the results are stated in the following Table as it regards the value of $\sqrt{k^2+h^2}$, and the rest in the first Table, p. 629. The times of oscillation, as calculated by formula (22.) and as determined by experiment, are stated in the second Table on that page.

	Angle of oscillation θ.	$1+f\theta$.	Value of $\sqrt{k^2+h^2}$ calculated in inches from time of				Mean value of $\sqrt{k^2+h^2}$ in each experiment.	Mean value of $\sqrt{k^2+h^2}$ in all the experiments.
			One oscillation.	Two oscillations.	Three oscillations.	Four oscillations.		
W=197·18 lbs.........	5°	1·0004762	11·336	11·780	12·356	12·461	11·983	12·033
	7 30′	1·0010723	12·318	12·200	12·828	12·903	12·562	
h=4·9 inches	10	1·0019072	12·003	11·620	11·620	11·870	11·550	
W=255·43 lbs.........	2 30	1·0000235	11·128	10·731	11·194	11·227	11·07	11·076
	5	1·0004762	9·429	10·328	10·924	11·123	10·451	
h=5·97 inches	7 30	1·0010723	11·910	11·315	11·976	11·935	11·784	
	10	1·0019072	10·332	10·818	11·446	11·414	11·000	